UNIT WITT VECTORS

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1. The multiplicative group W_n^* of units, in the ring W_n of Witt vectors [2] of length n with coefficients in an algebraically closed field k of characteristic $p>0$, is an algebraic group variety over k. It decomposes into the direct product of group varieties $W_n^*=W_1^* \times U_{n-1}$, where W_1^* is isomorphic to the one-dimensional multiplicative group G_m, and U_{n-1} is the algebraic subgroup defined by $x_0=1$; the decomposition is given by $(x_0, (1, x_1, \cdots, x_{n-1})) \to (x_0, x_0^p x_1, \cdots, x_0^{p^{n-1}} x_{n-1})$. Dieudonné has remarked [1] that U_{n-1} is isomorphic as *a formal Lie group* to the additive group W_{n-1}^+. We will show that these groups are isomorphic *as algebraic groups* only when $p \geqq 3$, and we will determine the structure of U_{n-1} in characteristic 2.

2. For the analytic assertions in this section, we refer to Hasse [2, §17]. The Witt vectors of infinite length form a complete discrete valuation ring W_∞ of characteristic 0, with residue field k, and maximal ideal generated by the prime p. Let

$$U^{(n)} = \{z \in W_\infty;\ z \equiv 1 \bmod p^n\}$$

so that $U^{(1)}/U^{(n+1)} = U_n$. Let $U_n^{(i)}$ be the subgroup $U^{(i)}/U^{(n+1)}$ of U_n. The series

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots$$

converges for all $(1+x) \in U^{(1)}$, and defines an analytic homomorphism from $U^{(n)}$ into the additive group $p^n W_\infty$ for all $n>0$. Hence it induces by passage to the quotient a homomorphism

$$\lambda_n\colon U_n \to pW_\infty/p^{n+1}W_\infty \cong W_n^+.$$

λ_n is a homomorphism of algebraic groups, i.e., is an everywhere defined rational mapping, since only a finite number of terms in the log series contribute to λ_n. We also have the usual exponential series $\exp x$, which converges for all x of order $>1/(p-1)$, and which defines an analytic homomorphism of $p^n W_\infty$ into $U^{(n)}$ inverse to the logarithm, provided $n>1/(p-1)$. Passage to the quotient yields a rational homomorphism

$$\epsilon_n^{(i)}\colon p^i W_\infty/p^{n+1} W_\infty \to U_n^{(i)}$$

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inverse to $\lambda_n | U_n^{(i)}$ for all $i > 1/(p-1)$. If $p \geqq 3$, the inequality $i > 1/(p-1)$ holds for all $i > 0$. *Thus λ_n is an isomorphism of the algebraic group U_n with W_n^+, whose inverse is $\epsilon_n^{(1)}$, if the characteristic of k is $\geqq 3$.*

3. Assume now $p=2$. Then the restriction of λ_n to the subgroup $U_n^{(2)}$ of U_n defined by $x_1=0$ is an isomorphism. We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_n^{(2)} & \longrightarrow & U_n & \xrightarrow{\pi} & G_a & \longrightarrow & 0 \\ & & \cong\downarrow & & \lambda_n\downarrow & & \downarrow\psi & & \\ 0 & \longrightarrow & W_{n-1}^+ & \xrightarrow{V} & W_n^+ & \xrightarrow{R^{n-1}} & G_a & \longrightarrow & 0 \end{array}$$

where $\pi(1, x_1, \cdots, x_n) = x_1$, $G_a = W_1^+$ is the one-dimensional additive group, and V, R denote the shift and restriction homomorphisms of Witt vectors. It is clear without introducing λ_1 that U_1 is isomorphic to G_a. For $n \geqq 2$, the class of U_n in $\mathrm{Ext}(G_a, W_{n-1}^+)$ will be completely determined by the rational endomorphism ψ of G_a (Serre [**3**, Chapter VII]). Putting $\lambda_n(1, x_1, \cdots, x_n) = (0, y_1, \cdots, y_n)$, we have $\psi(x_1) = y_1$. The first two terms in the log series are the only ones which can have order 1, and a computation of $x - x^2/2$ within W_∞ shows that $y_1 = x_1^2 + x_1$. λ_n is a separable isogeny of degree 2 whose kernel is $\{\pm 1\}$. The special property of the prime 2 here seems to be that the 2-adic logarithm of -1 is well-defined.

Note added in proof. J.-P. Serre has determined the structure of the *pro-algebraic group* $U^{(1)}$, using a different technique. Cf. his *Sur les corps locaux à corps résiduel algébriquement clos*, Bull. Soc. Math. France **89** (1961), §1.8, p. 115.

Bibliography

1. J. Dieudonné, *Lie groups and Lie hyperalgebras over a field of characteristic $p>0$. II*, Amer. J. Math. **77** (1955), 218–244.
2. H. Hasse, *Zahlentheorie*, Akademie-Verlag, Berlin, 1949.
3. J.-P. Serre, *Groupes algébriques et corps de classes*, Hermann, Paris, 1959.

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