ON APPROXIMATION BY FUNCTIONS OF LESSER VALENCE

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1. **Introduction.** Let G be a simply connected region in the plane of the complex variable z. The set of functions analytic in G and either identically constant or h-valent at most in G form a closed set H. That is to say, any limit in G (uniform on every closed set in G) of functions of the family also belongs to H [2, p. 8].

Let $w=f(z)$ be analytic and uniformly limited and k-valent in G. For every function $h(z)$ uniformly limited and of class H we set

$$\lambda = \text{l.u.b.}[\,|\,h(z) - f(z)\,|\,,\ z \text{ in } G].$$

Let $L(h)$ denote the greatest lower bound of all numbers λ. It follows by the theory of normal families [4, §§10, 18] that there exists at least one function in H for which $\lambda=L(h)$. Such an extremal function will be denoted by $h^*(z)$.

It will be understood throughout this paper that $h<k$. Then $L(h)$ must be positive. For if $L(h)$ were zero, there would exist a sequence $h_1(z), h_2(z), \cdots$ of functions in H whose corresponding λ-sequence $\lambda_1, \lambda_2, \cdots$ would converge to zero with the result that the sequence $h_1(z), h_2(z), \cdots$ would converge in G (uniformly on every closed set in G) to $f(z)$. But this is impossible since the limit of such a sequence is at most h-valent in G.

Walsh has proved [1, p. 345] that if $f(z)=z^k$, where k is a positive integer and G is the region $|z|<1$, then for every $h<k$ the value of $L(h)$ is unity. He has also found extremal functions $h^*(z)$ for such an $f(z)$ and such a region G. Our object in the present paper is to add a few results to those of Walsh. Our Theorem 1 is an appraisal on $L(h)$. In Theorem 2 the precise value of $L(h)$ is found for a restricted situation. Theorems 3 and 4 are initial results on the relation between the extremal functions and the character of the Riemann configuration R onto which G is mapped by $w=f(z)$.

2. **Radius of j-coverage.** Let $w=f(z)$ be analytic and uniformly limited and k-valent in a simply connected region G. Let R denote the Riemann configuration over the w-plane onto which G is mapped by $w=f(z)$. With every finite point w_0 of the w-plane we associate a non-negative real number $M_j(w_0)$ called the *radius of j-coverage* at w_0 (more exactly, the radius of precise j-coverage) as follows:

Received by the editors November 28, 1960 and, in revised form, January 18, 1961.

(a) if there exists a positive r such that for every point w in the region $|w-w_0|<r$ there are precisely j points of R having affix w (a branch point of R of order $p-1$ counting as p points of R), then we define $M_j(w_0)$ to be the largest r for which there is such precise j-coverage of the region $|w-w_0|<r$.

(b) if there exists no positive r such that for every w in the region $|w-w_0|<r$ there are precisely j points of R having affix w, we define $M_j(w_0)$ to be zero.

Verification of the statement that there is a largest r in (a) can be made in the manner indicated by Seidel and Walsh in their definition of radius of p-valence [**3**, p. 162]. It is to be noted, however, that while their radius of p-valence is associated with each point of the Riemann configuration R, our radius of j-coverage is associated with each finite point of the w-plane.

Let M_j denote the least upper bound for all $M_j(w)$. For each j there is at least one w at which $M_j(w)=M_j$. We denote by M_j^* the greatest number to be found among those M_i for which $i\geqq j$.

3. **Circular hull and circular kernels.** Let A denote a limited plane point set. The reader will readily verify the existence of a unique minimal closed circular region Q containing A. Q is the *circular hull* of A [**5**, p. 96].

If E is a limited plane point set having interior points, a maximal open circular region K contained in E is a *circular kernel* of E [**5**, p. 96]. A set E may have more than one circular kernel. The region bounded by two concentric circles has infinitely many circular kernels.

4. **Appraisal on $L(h)$.** Let $w=f(z)$ be analytic and uniformly limited and k-valent in a simply connected region G. Let R denote the Riemann configuration over the w-plane onto which G is mapped by the function $w=f(z)$. Let S denote the projection of R on the w-plane. That is to say, S shall denote the set of those points in the w-plane over each of which lies at least one point of R. Denote by S_j the set of points w having positive radius of j-coverage. Except for the trivial case where $f(z)$ is identically constant in G, the projection of R is a region (not necessarily simply connected). The radius of circular kernels of S_j is M_j. Finally, let r^* denote the radius of the circular hull of S. Then we have

THEOREM 1. *Let $w=f(z)$ be analytic and uniformly limited and k-valent in a simply connected region G. Then the measure $L(h)$ of best approximation to $f(z)$ in G by functions of the set H is bounded above* *by the radius r^* of the circular hull of S and is bounded below by the largest value to be found among the radii of circular kernels of the projections $S_{h+1}, S_{h+2}, \cdots, S_k$. That is to say, $M_{h+1}^* \leqq L(h) \leqq r^*$.*

PROOF. Let w^* denote the center of the circular hull of S. For the function $g(z) \equiv w^*$ of class H we have $\lambda = r^*$. Consequently $L(h)$ does not exceed r^*.

It remains to prove that $L(h)$ is not less than M_{h+1}^*. Suppose $L(h) = M' < M_{h+1}^*$. In accordance with the definition of M_{h+1}^* there exists a point w_0 together with a positive integer j_0 not less than $h+1$ such that $M_{j_0}(w_0) = M_{h+1}^*$. Choose a positive number M'' such that $M' < M'' < M_{h+1}^*$. Consider in G the locus $|f(z) - w_0| = M''$. The analysis set forth by Seidel and Walsh [3, §14] applied to the present situation shows that the locus $|f(z) - w_0| = M''$ consists of a set of Jordan curves J_i (one or more in number not exceeding j_0) such that the total number of zeros (counted according to multiplicity) of $f(z) - w_0$ enclosed by them is j_0. Let $h^*(z)$ be an extremal function of class H, that is, a function of the set H for which we have $\lambda = L(h) = M'$. If we write

$$Q(z) = h^*(z) - f(z) = \{h^*(z) - w_0\} - \{f(z) - w_0\},$$

we have $|Q(z)| < M''$ everywhere in G. Then on each Jordan curve J_i we have

$$|f(z) - w_0| = M'', \qquad |Q(z)| < M''.$$

It follows by Rouché's Theorem [1, p. 6] that the function $\{f(z) - w_0\} + Q(z)$ has precisely as many zeros interior to each J_i as has $f(z) - w_0$. This means that $h^*(z)$ takes the value w_0 at least $h+1$ times in G, contrary to the hypothesis that $h^*(z)$ belongs to the set H. The contradiction thus reached proves that $L(h)$ can not be less than M_{h+1}^*.

When $h=1$, the class H becomes the family of functions analytic and univalent in G; and Theorem 1 becomes a result in the problem of approximating to a multivalent function by univalent functions. Theorem 1 then has $L(1) \geqq M_2^*$. Moreover, when $k \geqq 2$, if we let q denote the largest of the integers $j = 2, 3, \cdots, k$ for which $M_j = M_2^*$, then the proof used in Theorem 1 shows that every function $F(z)$ analytic in G and such that

$$|F(z) - f(z)| \leqq \mu < M_2^*, \qquad z \text{ in } G,$$

is at least q-valent in G.

5. **Extremal functions.** Questions come to mind in connection with Theorem 1. What is the precise value of $L(h)$? When is there a unique

extremal function? When does the set of extremal functions include an identically constant function? How are the extremal functions $h^*(z)$ related to R and S and the S_j? The following three theorems provide a beginning toward the solutions of these problems. In Theorem 2 the term k-sheeted circle has the meaning given to it by Seidel and Walsh [3, p. 159].

THEOREM 2. *If the Riemann configuration R onto which G is mapped by $w=f(z)$ is a k-sheeted circle having center (or centers) of affix w_0, then $L(h)=r^*$; and one extremal function is $h^*(z)\equiv w_0$.*

PROOF. The circular hull of S is identical with the closure of S; and we have $M_1^*=M_2^*= \cdots =M_k^*=r^*$. Consequently, $L(h)=r^*$; and one $h^*(z)$ is the constant function $h^*(z)\equiv w_0$.

THEOREM 3. *If there exists a function $f(z)$ for which $L(h)$ is less than r^*, then none of the extremal functions $h^*(z)$ is constant.*

PROOF. The existence of an identically constant function for which $\lambda=L(h)<r^*$ would require that S have a circular hull of radius less than r^*.

THEOREM 4. *If $f(z)$ is such that $L(h)=r^*$, then the set of extremal functions $h^*(z)$ includes just one identically constant function $h^*(z) \equiv w_H$, where w_H is the center of the circular hull of S. Conversely, if the set of extremal functions $h^*(z)$ includes an identically constant function, then $L(h)=r^*$ and the constant member of the set of extremal functions is unique and is $h^*(z)\equiv w_H$.*

PROOF. When $L(h)=r^*$, then as in Theorem 2 the function $h(z) \equiv w_H$ is an extremal function $h^*(z)$. If there were another identically constant extremal $h^*(z)\equiv w_Q\neq w_H$, then the projection S would have to be contained in both the regions $|w-w_H|<r^*$ and $|w-w_Q|<r^*$. This would lead to the conclusion that S must have a circular hull of radius less than r^*, contrary to hypothesis.

Conversely, if the set of extremals $h^*(z)$ contains an identically constant function $h^*(z)\equiv w_0$, then $L(h)$ can not be less than r^* by Theorem 3. Then by the first part of the present theorem it follows that the function $h^*(z)\equiv w_0$ is the unique constant member of the set of extremals.

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