A SIMPLE PROOF OF FROBENIUS'S INTEGRATION THEOREM

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The connection between Stokes's Integral Theorem and the Frobenius-Cartan Integration Theorem concerning Pfaffian systems has been noted a long time. In this note, we generalize Stokes's theorem to implicit vector valued differential forms and derive from it a general Frobenius theorem concerning mappings in Banach spaces. The only difficulty in the proof arises in the need to show differentiability with respect to a parameter of solutions of a certain differential equation, but is is easily overcome. The generality of the theorem seems to be necessary for applications to the new subjects of infinite groups and of differential geometry in infinitely many dimensions. E.g., it allows us to associate a local group to any infinite-dimensional Lie algebra in a Banach space. For finite dimensional vector spaces we obtain the classical theorem with nearly minimal differentiability conditions [**4**]. Also for finite dimensional spaces, one might derive from it parts of the Cartan-Kähler theory of integral manifolds [**3**] for not completely integrable C^∞ systems.

1. All spaces in this note are real or complex Banach spaces. The only topology to be considered is the norm topology and the topologies induced by it in the spaces of linear mappings. A *mapping* will always be a bounded linear transformation of a Banach space into another one, a *function* is a continuous map of spaces. Given two spaces E, F with neighborhoods $U\subset E$, $V\subset F$, a *differential form* is a function $A(x, y)\colon U\times V\to \mathfrak{L}(E, F)$, taking values in the space of all mappings of E into F. We will denote by $k=A(x, y)h$, $h\in E$, $k\in F$ the image of h under the mapping, image of (x, y).

A function $f(x)\colon U\to F$ is said to be (Fréchet) differentiable in $x_0\in U$, if $df(x_0)h=\lim_{\lambda\to 0}(1/\lambda)(f(x_0+\lambda h)-f(x_0))$ defines a *mapping* of E into F, and if furthermore there exists for every neighborhood V of the zero of F an $\epsilon(V)>0$ such that

$$f(x_0+h)-f(x_0)-df(x_0)h\in \|h\|V$$

for all h satisfying $0<\|h\|<\epsilon(V)$, $x_0+h\in U$. Frobenius's problem may then be stated as follows: *Given a differential form $A(x, y)\colon U\times V\to\mathfrak{L}(E, F)$, find a function $f(x)\colon U\to F$ such that $df(x)=A(x, f(x))$, under the initial condition $f(x_0)=y_0$, $x_0\in U$, $y_0\in V$.*

The Fréchet differential is a straightforward generalization of the

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directional derivative. For finite dimensional spaces, $A(x, y)$ is a vector valued differential form, or a system (ω^i) of Pfaffian forms. Our problem then is to find a vector of functions $f^i(x', \cdots, x^n)$ such that $df^i - \omega^i = 0$ under given initial conditions and that the Jacobian $\partial(f^i)/\partial(x^j)$ be of maximum rank.

A differential form may itself be differentiable. Its differential may be made explicit by the partial differentials d_x and d_y, operating on E and F respectively:

$$[dA(x, y)h]k \times l = [d_xA(x, y)h]k + [d_yA(x, y)h]l, \quad h, k \in E, l \in F.$$

If y is given by a Pfaffian equation $dy = A(x, y)$, the substitution $l = dy(x)k = A(x, y)k$ induces from dA a function $U \times V \to \mathfrak{L}(E \wedge E, F)$, the *formal exterior differential* (in E. Cartan's terminology: the exterior differential mod $dy = A(x, y)$ [2]):

$$\begin{aligned}\delta A(x, y)(h \wedge k) = [d_xA(x, y)h]k &+ [d_yA(x, y)h]A(x, y)k \\ &- [d_xA(x, y)k]h - [d_yA(x, y)k]A(x, y)h.\end{aligned}$$

Frobenius's problem is then solved by the

THEOREM. *Assume $dA(x, y)$ to be a bounded continuous function $\overline{U} \times \overline{V} \to \mathfrak{L}(E, \mathfrak{L}(E, F))$. The equation*

$$dy = A(x, y) \tag{1}$$

subject to the initial condition

$$y(x_0) = y_0, \qquad x_0 \in U, \; y_0 \in V \tag{2}$$

has a unique continuously differentiable solution in some neighborhood of x_0 if and only if $\delta A(x, y) = 0$ in $U \times V$.

For functions $f: E \to F$ we may write $df = \delta f$, since $E = \otimes^1 E = \wedge^1 E$. Under our conditions a symmetry property holds for second differentials which implies $\delta\delta f = 0$. This takes care of the necessity part of the theorem, if it is assumed that a solution exists for arbitrary initial data.

2. A curve is a function $c: I \to E$ defined on the unit interval $I = [0, 1]$. We will denote by c_t its restriction to $I_t = [0, t]$, $0 \leqq t \leqq 1$. All curves considered in the sequel are C^1, hence rectifiable. Given a differential form $B(x): U \to \mathfrak{L}(E, F)$ and a curve $c: I \to U$, $U \subset E$, the integral of B on c_t is the integral on I_t

$$\int_{c_t} B(x)dx = \int_0^t B(c(\tau))c'(\tau)d\tau.$$

Here and in the future we will restrict ourselves to *convex* (e.g., spherical) neighborhoods. Any two points x_0, x_1 in U may then be joined by a C^1 curve in U, e.g., the segment $c(t)=(1-t)x_0+tx_1$. In order to solve (1) we first join x_0 to a nearby x_1 by a smooth curve c and solve the integral equation

$$
\begin{aligned}
y(t) &= y_0 + \int_{c_t} A(x, y(x))dx \\
&= y_0 + \int_0^t A(c(\tau), y(\tau))c'(\tau)d\tau.
\end{aligned} \tag{3}
$$

By our assumption the Fréchet differential $d_yA(x, y)$ is a *bounded* linear operator, hence $A(x, y)$ satisfies a Lipschitz condition in y. The usual successive approximations [1] assure us that (3) has a unique solution for x_1 in some convex neighborhood $U(x_0)$, and that $dy(t)dx = A(c(t), y(t))dx$ holds for $dx=c'(t)dt$ and all $t\in I$.

In order to prove our theorem, we will join x_0 and x_1 by any other smooth curve c^* in U. Let $y^*(t)$ be the solution of (3) corresponding to c^*. Under the conditions of our theorem, we show that $y(1)=y^*(1)$, i.e. $y(x)$ is uniquely defined by integration along smooth curves. Finally, we have to show that (1) holds for this $y(x)$ for *all* $h\in E$. In view of the last sentence of the preceding paragraph this will be established if for any $h\in E$, $\|h\|<\epsilon$, we find a c with $c(0)=x_0$, $c(1)=x_1$, $c'(1)=h$. Take in U a spherical neighborhood about x_1, with radius $\epsilon/2$. Then for $\|h\|<\epsilon$ the point $x_1-h/2$ is in that neighborhood, hence the "parabola" $c(t) = (1-t^2)x_0 + 2t(1-t)(x_1-h/2) + t^2x_1 = (1-t)\{(1-t)x_0+t(x_1-h/2)\} + t(1-t)(x_1-h/2)+tx_1\}$ is in U; this is a curve of the desired property.

3. Let c and c^* be two smooth curves joining x_0 and x_1 in U. A homotopy of c and c^* is defined by

$$c_s(t) = (1-s)c(t) + sc^*(t), \qquad s\in I. \tag{4}$$

All $c_s(t)$ are C^1 curves, for fixed s. By hypothesis, $\sup_{x\in\bar U, y\in\bar V}\|A(x, y)\| < \infty$, hence (3) has a solution for any c_s and some δ, $\|x_1-x_0\|<\delta$ (cf. [1, Theorem 1]). As a preliminary step, we have to study the differentiability with respect to s of this $y_s(t)$, given by

$$y_s(t) = y_0 + \int_0^t A(c_s(\tau), y_s(\tau))((1-s)c'(\tau) + sc^{*\prime}(\tau))d\tau. \tag{5}$$

If $dy_s(t)/ds$ exists and is continuous on the compact set $I\times I$, the function $z_s(t)=dy_s(t)/ds$ satisfies the integral equation

$$
\begin{aligned}
z_s(t) = &\int_0^t [d_x A(c_s(\tau), y_s(\tau))((1-s)c'(\tau) + sc^{*\prime}(\tau))](c^*(\tau) - c(\tau))d\tau \\
&+ \int_0^t A(c_s(\tau), y_s(\tau))(c^{*\prime}(\tau) - c'(\tau))d\tau \\
&+ \int_0^t [d_y A(c_s(\tau), y_s(\tau))((1-s)c'(\tau) + sc^{*\prime}(\tau))]z_s(\tau)d\tau,
\end{aligned} \tag{6}
$$

which is of the type

$$
z_s(t) = P(s, t) + \int_0^t Q(s, \tau)z_s(\tau)d\tau \tag{6a}
$$

with continuous (hence bounded) $P(s, t)$ and $Q(s, t)$ $((s, t)\in I\times I)$. Both (5) and (6) have unique solutions which are continuous on the whole of $I\times I$ (use [1, Theorem 3]), hence we may represent $y_s(t)$ as a Stieltjes integral in the variable s; using the abbreviation introduced in (6a) we have

$$
y_s(t) - y_0(t) = \int_0^s d_s y_\sigma(t) = \int_0^s P(\sigma, t)d\sigma + \int_0^s \int_0^t Q(\sigma, \tau)d_s y_\sigma(\tau)d\tau.
$$

Introducing $Z(s, t) = \int_0^s z_\sigma(t)d\sigma + y_0(t) - y_s(t)$, we finally have an integro-differential equation

$$
Z(s, t) = \int_0^s \int_0^t Q(\sigma, \tau)d_s Z(\sigma, \tau)d\tau, \qquad Z(0, t) = 0. \tag{7}
$$

We say that $Z(s, t)$ is Lipschitzian in s. For the integral this is trivial; to see it for $y_s(t)$ remark that $A(c_s(t), y_s(t))$ is uniformly continuous on the compact set $(s, t)\in I\times I$. If we denote by $L(c)$ the length of the curve c, we see from (3) that

$$
\|y_s(t) - y_{\hat{s}}(t)\| < (\|A\| + \epsilon)(L(c_t) + L(\overset{*}{c}_t))\,|s - \hat{s}|
$$

if for $\epsilon > 0$ we choose δ to have $|s-\hat{s}| < \delta$ imply $\|A(c_s(t), y_s(t)) - A(c_{\hat{s}}(t), y_{\hat{s}}(t))\| < \epsilon$. We know now that there exists a constant Z, $\|Z(s, t) - Z(\hat{s}, t)\| < Z|s-\hat{s}|$ for $|s-\hat{s}| < \delta$. This is sufficient to use the standard procedure to infer that $Z(s, t) = 0$ is the only solution of (7), as we have by successive approximations $\|Z(s, t)\| < \|Q\|Zst$, hence $\|Z(s, t)\| < 2^{k-1}\|Q\|^k Zst^k/k!$ for all natural k. Therefore $y_s(t)$ is a C^1 function in s.

4. By partial integration of the second term in (6) we obtain

$$\frac{dy_s(1)}{ds} = \int_0^1 [d_x A(c_s(\tau), y_s(\tau))\{(1-s)c'(\tau) + sc^{*\prime}(\tau)\}](c^*(\tau) - c(\tau))d\tau$$
$$- \int_0^1 [d_x A(c_s(\tau), y_s(\tau))(c^*(\tau) - c(\tau))]\{(1-s)c'(\tau) + sc^{*\prime}(\tau)\}d\tau$$
$$+ \int_0^1 [d_y A(c_s(\tau), y_s(\tau))\{(1-s)c'(\tau) + sc^{*\prime}(\tau)\}] \frac{\partial y_s(\tau)}{\partial s} d\tau$$
$$- \int_0^1 [d_y A(c_s(\tau), y_s(\tau))(c^*(\tau) - c(\tau))] \frac{\partial y_s(\tau)}{\partial t} d\tau.$$

As $dx = \{(1-s)c'(t) + sc^{*\prime}(t)\}dt + \{c^*(t) - c(t)\}ds$, $dy = (\partial y_s(t)/\partial t)dt + (\partial y_s(t)/\partial s)ds$, we finally have our Stokes's formula

$$\begin{aligned} y_1(1) - y_0(1) &= \int_0^1 \frac{dy_s(1)}{ds} ds \\ &= \iint_{I\times I} \{d_x A(x, y)dx \wedge dx + d_y A(x, y)dx \wedge dy \\ &= \iint_{I\times I} \delta A(x, y)dx, \end{aligned}$$

which from (5) may be written in an easily understood shorthand

$$\oint_{c^*-c} A(x, y)dx = \iint_{c(s,t)} \delta A(x, y)dx \tag{8}$$

and from which by the assumption of our theorem we have

$$y_1(1) = y_0(1),$$

or, in our previous notation $y(x_1) = y^*(x_1)$. This completes the proof.

In the finite dimensional case,

$$\dim y(U) = \min(\dim E, \ \min_{x\in U} \dim A(x, F)).$$

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