

A THEOREM ON CONVEX SURFACES

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In a previous paper [1] we presented a characterization of homothetic transformations between closed surfaces in terms of their mean curvatures. Here we reach a corresponding result about the Gaussian curvatures, which improves a theorem of A. Aeppli in the three-dimensional case [2, Satz 10]. Notations in [1] will be adopted.

LEMMA. Suppose S and \bar{S} are closed orientable strictly convex C^2 surfaces and $h: S \rightarrow \bar{S}$ is an order-preserving differentiable homeomorphism. If, referring to a common interior point as origin, $\bar{X} = kX$ and $k\bar{h}_{ij}h^{ij}g^{1/2}/\bar{g}^{1/2} \geq 2$, where h_{ij} and \bar{h}_{ij} are the second fundamental tensors, $(h^{ij}) = (h_{ij})^{-1}$ and g, \bar{g} are the determinants of the first fundamental tensors, then h is a homothetic transformation.

PROOF. Differentiating $\bar{X} = kX$,

$$\begin{aligned} \bar{X}_i &= kX_i + k_iX, \\ (1) \quad \bar{X}_{ij} &= kX_{ij} + k_iX_j + k_jX_i + k_{ij}X, \\ \bar{h}_{ij}\bar{N} + \Gamma_{ij}^h\bar{X}_h &= k(h_{ij}N + \Gamma_{ij}^hX_h) + k_iX_j + k_jX_i + k_{ij}X. \end{aligned}$$

Taking scalar product with N ,

$$(2) \quad \bar{h}_{ij}(N \cdot \bar{N}) - p\Gamma_{ij}^h k_h = kh_{ij} - pk_{ij}.$$

From (1) we have

$$\bar{g}^{1/2}\bar{N} = k^2g^{1/2}N + kX \times (k_1X_2 - k_2X_1).$$

Taking scalar product with N ,

$$\bar{g}^{1/2}(N \cdot \bar{N}) = k^2g^{1/2} + N \cdot [kX \times (k_1X_2 - k_2X_1)].$$

Substituting in (2) and contracting with h^{ij} ,

$$h^{ij}k_{ij} + C^h k_h = \frac{k}{p} (2 - k\bar{h}_{ij}h^{ij}g^{1/2}/\bar{g}^{1/2}) \leq 0.$$

Hence $k = \text{const}$ [3; 4, Chapter II]. And so h is a homothetic transformation.

THEOREM. Given two closed orientable strictly convex surfaces S, \bar{S} in E^3 of class C^2 and a differentiable homeomorphism $h: S \rightarrow \bar{S}$, if,

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referring to an origin interior to both surfaces, $\bar{X} = kX$, $k^2\bar{K} = K$, then h is a homothetic transformation with center O .

PROOF. We may assume h to be orientation preserving for if it reverses the orientation then we may combine to it the reflexion $\bar{X} \rightarrow -\bar{X}$. By well-known principles we have $k\bar{h}_{ij}h_{ij}g^{1/2}/\bar{g}^{1/2} \geq 2$, since $|h_{ij}| = |k\bar{h}_{ij}g^{1/2}/\bar{g}^{1/2}|$. Hence h is a homothetic transformation, by the lemma.

COROLLARY 1. *With the same assumption except that*

$$k^2\bar{K} = K + 4(X \cdot N / X \cdot X)H + 4(X \cdot N / X \cdot X)^2$$

then h is an inversion with center O .

PROOF. Let $h^*: S \rightarrow S^*$ be the inversion $X^* = X / X \cdot X$. Then

$$(X^* \cdot X^* / X \cdot X)K^* = K + 4(X \cdot N / X \cdot X)H + 4(X \cdot N / X \cdot X)^2.$$

And by the main theorem $h(h^*)^{-1}: S^* \rightarrow \bar{S}$ is a homothetic transformation. Hence h is an inversion.

COROLLARY 2 (C. S. HSÜ [5]). *If the Gaussian curvature of a closed surface at each point equals the inverse square distance from an interior point O , then it is a sphere with center O .*

REMARK 1. Under suitable boundary condition, the theorem holds for surfaces with boundaries. In this case we use the result of [6] in place of [3].

REMARK 2. Following some limiting process the theorem can be proved, when O lies on both surfaces. In this case too we use [6].

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