A CHARACTERIZATION OF MONOMIALS

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A monomial of n complex variables is a function of the form

$$az_1^{p_1} z_2^{p_2} \cdots z_n^{p_n}$$

where $p_1, \cdots, p_n$ are non-negative integers and where a is constant. The set

$$E = \{(z_1, \cdots, z_n) \mid |z_1| < 1, \cdots, |z_n| < 1\}$$

is called the unit polycylinder. The set

$$D = \{(z_1, \cdots, z_n) \mid |z_1| = 1, \cdots, |z_n| = 1\}$$

is said to be the distinguished boundary of E. Note that D is not the whole boundary of E.

We want to prove the following theorem:

The monomials are the only entire functions whose absolute value is constant on the distinguished boundary of the unit polycylinder.

PROOF. Denote by $\mathbf{C}^n$ the space of n complex variables. The elements are the vectors

$$z = (z_1, \cdots, z_n)$$

whose coordinates z_ν are complex numbers.

Now, let f be an entire function, whose absolute value $|f|$ is constant on D. If $|f|$ is identically zero on D, then f is identically zero on $\mathbf{C}^n$. Therefore, f is a monomial.

We may exclude this case and assume, without loss of generality, that $|f(z)| = 1$ for $z \in D$. Now, we want to show that such an entire function f is either constant or has zeros in $\overline{E}$. Assume $f(z) \neq 0$ for $z \in \overline{E}$. Then, we have by a well-known theorem

$$\max_{z \in \overline{E}} |f(z)| = \max_{z \in D} |f(z)| = 1,$$

$$\min_{z \in \overline{E}} |f(z)| = \min_{z \in D} |f(z)| = 1.$$

Therefore, f is constant. Consequently, f is either constant or has zeros in E.

Now, we want to prove that $f(z) \neq 0$ for all z in

$$A = \{(z_1, \cdots, z_n) \mid z_1 z_2 \cdots z_n \neq 0\}.$$

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Define the function g by

$$g(z_1, \cdots, z_n) = \overline{f\left(\frac{1}{\bar{z}_1}, \cdots, \frac{1}{\bar{z}_n}\right)}$$

on A. This function is holomorphic on A since its real partial derivatives exist, are continuous, and satisfy the Cauchy-Riemann equations

$$g_{\bar{z}_\nu}(z_1, \cdots, z_n) = \overline{f_{z_\nu}\left(\frac{1}{\bar{z}_1}, \cdots, \frac{1}{\bar{z}_n}\right)\frac{\partial}{\partial z_\nu}\left(\frac{1}{\bar{z}_\nu}\right)} = 0.$$

For $z \in D \subset A$, we have

$$g(z_1, \cdots, z_n) = \overline{f\left(\frac{1}{\bar{z}_1}, \cdots, \frac{1}{\bar{z}_n}\right)} = \overline{f(z_1, \cdots, z_n)}$$
$$= \frac{1}{f(z_1, \cdots, z_n)}.$$

Therefore

$$g(z_1, \cdots, z_n) \cdot f(z_1, \cdots \cdot z_n) = 1 \qquad \text{for } (z_1, \cdots, z_n) \in D.$$

The function $g \cdot f$ is holomorphic on A and identically one on D. Since A is a connected, open neighborhood of D, the function $g \cdot f$ is equal to one on A:

$$f(z) \cdot g(z) = 1 \qquad \text{for } z \in A.$$

Therefore, we have $f(z) \neq 0$ for $z \in A$.

Our function f vanishes at most on the planes $\{(z_1, \cdots, z_n) \mid z_\nu = 0\}$. Therefore f has the form[1]

$$f(z_1, \cdots, z_n) = z_1^{p_1} \cdot z_2^{p_2} \cdots z_n^{p_n} \cdot h(z_1, \cdots, z_n)$$

where $p_1, \cdots, p_n$ are non-negative integers and h is an entire function which does not vanish at all, and whose absolute value is constant on D. Consequently, h is a constant. We obtain

$$f(z_1, \cdots, z_n) = a z_1^{p_1} \cdots z_n^{p_n}$$

q.e.d.

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[1] See: Osgood, *Lehrbuch der Funktionentheorie*, Vol. III, Chapter III, §26.