ENTIRE FUNCTIONS IN SEVERAL VARIABLES WITH CONSTANT ABSOLUTE VALUES ON A CIRCULAR UNIQUENESS SET[1]

S. BOCHNER

If for a function $\phi(z)=\sum_0^\infty a_p z^p$ in the entire z-plane we have $|\phi(z)|=1$ on $|z|=1$, then the product

$$\sum_0^\infty a_p z^p \cdot \sum_0^\infty \frac{\bar{a}_q}{z^q}$$

which is analytic in $z\neq 0$ has value 1 on $|z|=1$. Therefore it is identically 1, and thus $\phi(z)\neq 0$ for $z\neq 0$. Therefore $\phi(z)=z^p h(z)$ where $h(z)\neq 0$ everywhere. But again $|h(z)|=1$ on $|z|=1$, and for such an $h(z)$ we have $h(z)=c$, so that $\phi(z)=cz^p$. By the use of the same method, Bojanic and Stoll [1] have recently given the following generalization to functions which are holomorphic in the entire $\mathbf{C}^n$, for any n.

THEOREM 1. *If for an entire function* $f(z)\equiv f(z_1,\cdots,z_n)$ *we have*

$$|f(\zeta_1,\cdots,\zeta_n)| = 1$$

on the set

$$(1) \qquad |\zeta_1| = 1,\cdots, |\zeta_n| = 1$$

then

$$f(z_1,\cdots,z_n) = c\, z_1^{p_1}\cdots z_n^{p_n}.$$

The authors invoke the lemma that if an entire function is $\neq 0$ for $z_1\cdots z_n\neq 0$ then it can be represented as a product

$$c\, z_1^{p_1}\cdots z_n^{p_n} h(z)$$

in which $h(z)\neq 0$ everywhere. We propose to avoid taking recourse to this lemma and to obtain a more systematic theorem in the process.

We replace the point set (1) by a general point set S having the following properties

(i) S is circular, that is, if $(\zeta_1,\cdots,\zeta_n)\in S$, then also $(\zeta_1 t,\cdots,\zeta_n t)\in S$ for any $|t|=1$.

(ii) S is connected.

(iii) S is a uniqueness set for entire functions, in the sense that if

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$f(z)$ is 0 on S it is $\equiv 0$. We note that S is of this kind if it is part of the boundary of a domain R such that, for z in R, $f(z)$ can be represented by a suitable Cauchy integral in which integration extends over S only.

THEOREM 2. *If for an entire function we have* $|f(\zeta)|=1$ *on a set* S *then* $f(z)$ *is a homogeneous polynomial*

$$A(z_1, \cdots, z_n) = \sum_{p_1+\cdots+p_n=p} a_{p_1\cdots p_n} z_1^{p_1} \cdots z_n^{p_n} \tag{2}$$

of some finite degree $p \geqq 0$.

PROOF. For any fixed $(\zeta) \in S$ we form the function

$$\phi(z) = f(\zeta_1 z, \cdots, \zeta_n z)$$

in $\mathbf{C}^1$. By property (i) of S we have $|\phi(z)|=1$ on $|z|=1$, and hence by our introductory statement we have $\phi(z)=c\, z^p$, that is

$$f(\zeta_1 z, \cdots, \zeta_n z) = z^p f(\zeta_1, \cdots, \zeta_n).$$

Up to here the exponent p is a function of ζ. However it follows from

$$|z|^p = |f(\zeta_1 z, \cdots, \zeta_n z)|$$

(for $z=2$, say) that p is a continuous function on S. Furthermore it is integer-valued, and S is connected by property (ii). Therefore p is a constant number.

We can now form the difference

$$f(w_1 z, \cdots, w_n z) - z^p f(w_1, \cdots, w_n).$$

It is an entire function in $w_1, \cdots, w_n$ and z, and it is 0 for $(w) \in S$. By property (iii) it is $\equiv 0$. If we now introduce the power series for $f(z)$, the conclusion of Theorem 2 follows.

We next add one further property of S which is much more structural than the preceding ones.

(iv) On S we have

$$\bar{\zeta}_\nu = \frac{\lambda_\nu(\zeta)}{D(\zeta)}, \qquad \nu = 1, \cdots, n,$$

where $\lambda_1(z), \cdots, \lambda_n(z)$, $D(z)$ are (homogeneous) polynomials in $z_1, \cdots, z_n$.

If now we introduce the factorization into irreducible polynomials

$$D(z) = D_1(z)^{q_1} \cdots D_m(z)^{q_m} \tag{3}$$

then the following conclusion results.

THEOREM 3. *Furthermore, we have*

$$f(z_1, \cdots, z_n) = c\, D_1(z)^{p_1} \cdots D_m(z)^{p_m} \tag{4}$$

for some exponents $p_1 \geqq 0, \cdots, p_m \geqq 0$.

PROOF. If with the coefficients of (2) we form the polynomial

$$B(z_1, \cdots, z_n) = \sum_{p_1 + \cdots + p_n = p} \overline{a_{p_1 \cdots p_n}} \lambda_1^{p_1} \cdots \lambda_n^{p_n}$$

then our assumption

$$|f(\zeta)|^2 = 1 = f(\zeta)\overline{f(\zeta)}$$

implies that we have

$$A(z) \cdot B(z) = D(z)^p \tag{5}$$

on S. Using again (iii) we conclude that this holds identically in z. But all factors in (5) are polynomials and therefore (5) implies (4) by simple algebra.

REMARK. If also $|D_\nu(\zeta)| = 1$, $\nu = 1, \cdots, n$, then conversely every function (4) has constant absolute value on S.

Theorem 1 subsumes under Theorem 3 if we put

$$D(z) = z_1 \cdots z_n; \qquad \lambda_\nu(z) = \frac{D(z)}{z_\nu}, \qquad \nu = 1, \cdots, n.$$

But we also obtain interesting statements for some types of symmetric domains. Assume for instance that $n = k^2$ and that our variables $z_1, \cdots, z_n$ constitute a square array $\{z_{pq}\}$, $p, q = 1, \cdots, k$. The associated "natural" uniqueness set is formed by the unitary matrices

$$\sum_{r=1}^{k} \zeta_{pr} \bar{\zeta}_{qr} = \delta_{pq}, \tag{6}$$

see [2].

This gives

$$\bar{\zeta}_{\mu\nu} = \frac{\lambda_{\mu\nu}(\zeta)}{\det|\zeta|}$$

where $\lambda_{\mu\nu}$ are certain minors of the matrix $\{\zeta_{pq}\}$ and

$$D(\zeta) = \det|\zeta_{pq}|$$

is its determinant. Now, the determinant is an irreducible polynomial and $|D(\zeta)|=1$. Hence the following theorem.

THEOREM 4. *If $f(z_{rs})$ is defined holomorphic over the entire matrix space, then it has absolute value* 1 *on the unitary set* (6) *if and only if*

$$f(z) = e^{i\alpha}(\det|z_{rs}|)^p$$

for some integer $p \geqq 0$.

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PRINCETON UNIVERSITY

ON A CRITERION FOR DETERMINATE MOMENT SEQUENCES

DAVID S. GREENSTEIN

On page 20 of [**4**], the following criterion is given as sufficient for the determinacy of a Hamburger moment sequence $\{\mu_n\}$:

$$\liminf \left(\mu_{2n}^{1/2n}/n^2\right) < \infty. \tag{1}$$

Attributed to Perron [**2**], it is obtainable only by transforming a criterion for Stieltjes determinacy due to Perron. In doing so, I find (1) not to follow from Perron's result. In this note, I shall make the proper correction to eliminate confusion caused by the error (e.g., (1) if valid would be more general than Carleman's well known criterion [**1**]). I also give an example to show that (1) is invalid.

Symmetrization of all mass distributions with the moments μ_n shows that $\{\mu_n\}$ is determinate provided that there is no more than one symmetric distribution with the moments $\mu_0, 0, \mu_2, 0, \cdots$. But the latter condition is easily shown to be equivalent to Stieltjes determinacy of the moment sequence $\{\mu_{2n}\}$ (not the same as Hamburger determinacy of $\{\mu_{2n}\}$ [**4**]).

Perron [**2**] gives as a sufficient condition for Stieltjes determinacy of $\{\mu_{2n}\}$

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