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ON A RESULT OF BAER

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The theorem is that *if H and N are subgroups of the group G, with N normal in G, and if the set of commutators*

$$\{hnh^{-1}n^{-1} \mid h \in H, n \in N\}$$

is finite, then so is the group $[H, N]$ generated by these. Here we give a proof that seems considerably simpler than the original one [**1**] (cf. also [**2**, exposé 3] for the case where both H and N are normal in G).

The most important special case of Baer's result concerns a group G whose center is of finite index, in which case the assertion is that the commutator group of G is finite. Here is a brief proof, similar to the one given in [**3**, p. 59]: It suffices to show that any product of commutators of elements of G can be written as such a product with at most n^3 factors, n being the index of the center of G. Noting that there are at most n^2 distinct commutators, and that in any product of commutators any two factors may be brought together by replacing the intermediate factors by conjugates, also commutators, it suffices to show that the $(n+1)$th power of a commutator is the product of n

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commutators. But if $a, b \in G$, then $(aba^{-1}b^{-1})^n$ is central, so

$$(aba^{-1}b^{-1})^{n+1} = b^{-1}(aba^{-1}b^{-1})^n b(aba^{-1}b^{-1}),$$

which may be written

$$b^{-1}((aba^{-1}b^{-1})^{n-1}(ab^2a^{-1}b^{-2}))b,$$

a product of n commutators.

We proceed to prove the general result. It is worth remarking that if one is only interested in the case where both H and N are normal, the trickiest points below collapse to trivialities.

Assume, as we may, that $G = HN$, and consider the set S of all commutators of conjugates of elements of H by elements of N. Any conjugate of an element of H is of the form nhn^{-1}, with $n \in N$, $h \in H$, so each element of S is of the form

$$(nhn^{-1})n_1(nhn^{-1})^{-1}n_1^{-1} = (hnh^{-1}n^{-1})^{-1}(h(n_1n)h^{-1}(n_1n)^{-1}),$$

with $n_1 \in N$, which shows that S is a finite subset of $[H, N]$. But S clearly generates $[H, N]$, and each inner automorphism of G permutes the elements of S. We deduce that *$[H, N]$ is normal in G*, and also that there exists a normal subgroup G_0 of G of finite index that centralizes S, hence also $[H, N]$. Now $G_0 \cap [H, N]$ is a central subgroup of $[H, N]$ of finite index, so $[[H, N], [H, N]]$ is finite. Since the latter subgroup is normal in G, we may divide by it to *suppose that $[H, N]$ is commutative.*

We now claim that the subgroup $[H, [H, N]]$ of $[H, N]$ is normal in G. Conjugation by elements of H clearly leaves it invariant, so we must show that if $n \in N$, $h \in H$, $m \in [H, N]$, then $n(hmh^{-1}m^{-1})n^{-1} \in [H, [H, N]]$. But the latter element can also be written

$$hn(n^{-1}h^{-1}nh)mh^{-1}m^{-1}n^{-1},$$

which, by the commutativity of $[H, N]$, is equal to

$$hnm(n^{-1}h^{-1}nh)h^{-1}m^{-1}n^{-1} = h(nmn^{-1})h^{-1}(nmn^{-1})^{-1} \in [H, [H, N]].$$

Note also that any commutator of H and $[H, N]$ is one of H and N, so there are only a finite number of such and they all commute. Furthermore, if we square any such commutator, say $hmh^{-1}m^{-1}$, we get $(hmh^{-1}m^{-1})^2 = (hmh^{-1})^2m^{-2} = hm^2h^{-1}m^{-2}$, which is also a commutator. Thus $[H, [H, N]]$ is finite. Dividing G by this subgroup, we see that we may *suppose that H centralizes $[H, N]$.*

To finish the proof, recall that $[H, N]$ is commutative and generated by a finite number of commutators $hnh^{-1}n^{-1}$, and note that here too the square of such a commutator is also a commutator:

$$(hnh^{-1}n^{-1})^2 = (hnh^{-1}n^{-1})(nh^{-1}n^{-1}h) = hnh^{-2}n^{-1}h = h^2nh^{-2}n^{-1}.$$

Thus $[H, N]$ is finite.

The most important application of Baer's result is to the proof of the nontrivial part of the following theorem: *If H and N are algebraic subgroups of the algebraic group G, and if at least one of H, N is connected, or normal in G, then $[H, N]$ is an algebraic subgroup of G.*

For the proof, first suppose N connected and let $H_0, H_1, \cdots, H_r$ be the components of H. For $i=0, \cdots, r$, the image of $H_i \times N$ under the rational map $(h_i, n) \to h_i n h_i^{-1} n^{-1}$ contains an open subset of its closure, which is a subvariety of G containing e; by general principles, the group generated by these images, i.e. $[H, N]$, is algebraic. Finally let N be normal in G, and suppose, as we may, that $G=HN$. If H_0, N_0 are the components of the identity of H, N respectively, then $[H_0, N]$ and $[H, N_0]$ are connected algebraic subgroups of G, hence so are all their conjugates, hence so also is the group Γ generated by all these. The algebraic group Γ is the smallest normal subgroup of G that contains $[H_0, N] \cup [H, N_0]$, and clearly $\Gamma \subset [H, N]$. Dividing G by Γ, we get the same situation, but with $[H_0, N] = [H, N_0] = e$. But in this case the set of commutators $hnh^{-1}n^{-1}$ is finite, so $[H, N]$ is finite.

There are easy counterexamples when both of the conditions in the previous result are dropped: e.g. take $G = GL(2, \mathbf{C})$, and let H, N be the subgroups of order two with generators

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \text{ respectively.}$$

An interesting consequence of the previous result is that *if H is an algebraic subgroup of the algebraic group G, then the smallest normal subgroup of G that contains H is algebraic.* For this subgroup is $H[H, G]$, which is algebraic since $[H, G]$ is algebraic and normal.

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