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COEFFICIENT ESTIMATES FOR UNIVALENT FUNCTIONS

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Let

$$(1) \quad f(z) = z + b_1/z + b_2/z^2 + \dots$$

be analytic and univalent in $|z| > 1$. It is an immediate consequence of the area theorem that $|b_1| \leq 1$. In 1938 Schiffer [5], using a variational method, and Golusin [2] independently derived the sharp inequality $|b_2| \leq 2/3$. It was then conjectured that $|b_n| \leq 2/(n+1)$, but this proved to be false without additional hypotheses. Indeed, Garabedian and Schiffer [1] found the true bound for $|b_n|$ to be $1/2 + e^{-6}$. A theorem of Golusin [2], however, asserts the following:

THEOREM. *Let $f(z)$ be analytic and univalent in $|z| > 1$ and have the expansion (1). For some $N \geq 1$, let $b_1 = b_2 = \dots = b_{N-1} = 0$. Then $|b_n| \leq 2/(n+1)$ for $n = N, N+1, \dots, 2N$. (In particular, $|b_2| \leq 2/3$.)*

Jenkins [4] has recently obtained this result by means of his General Coefficient Theorem. It is the purpose of the present note to give a simple proof. Clearly, it suffices to show $|b_{2N-1}| \leq 1/N$ and $|b_{2N}| \leq 2/(2N+1)$.

The n th Faber polynomial $F_n(w)$ associated with the function (1) is defined by

$$F_n[f(z)] = z^n + \sum_{m=1}^{\infty} \beta_{nm} z^{-m}.$$

The following inequalities, due to Grunsky [3], constitute an elementary generalization of the area theorem:

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$$\left| \sum_{n,m=1}^N m\beta_{nm}x_nx_m \right| \leq \sum_{n=1}^N n|x_n|^2$$

for each N an arbitrary complex $x_1, x_2, \dots, x_N$. In particular,

$$(2) \quad |\beta_{NN}| \leq 1.$$

Now suppose $b_1 = b_2 = \dots = b_{N-1} = 0$ in (1). Then

$$\begin{aligned} [f(z)]^N &= z^N [1 + b_N/z^{N+1} + b_{N+1}/z^{N+2} + \dots]^N \\ &= z^N \left\{ 1 + N(b_N/z^{N+1} + \dots) \right. \\ &\quad \left. + \frac{N(N-1)}{2!} (b_N/z^{N+1} + \dots)^2 + \dots \right\}. \end{aligned}$$

Hence $F_N[f(z)] = [f(z)]^N$, and $\beta_{NN} = Nb_{2N-1}$. Therefore, by (2), $|b_{2N-1}| \leq 1/N$.

To estimate $|b_{2N}|$, consider the function

$$\begin{aligned} g(z) &= (f(z^2))^{1/2} = z[1 + b_N/z^{2N+2} + b_{N+1}/z^{2N+4} + \dots]^{1/2} \\ &= z\{1 + 1/2(b_N/z^{2N+2} + \dots) - 1/8(b_N/z^{2N+2} + \dots)^2 + \dots\} \\ &= z + \sum_{j=1}^{\infty} c_j z^{-j}, \text{ say.} \end{aligned}$$

Here $c_j = 0$ for $j \leq 2N+1$, so $|c_{4N+1}| \leq 1/(2N+1)$, by what has just been proved. But $c_{4N+1} = (1/2)b_{2N}$, so $|b_{2N}| \leq 2/(2N+1)$, which finishes the proof.

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