REFERENCES

1. Z. Nehari, *On the singularities of Legendre expansions*, J. Rat. Mech. Anal. 5 (1956), 987-992.
2. S. Bergman, *Zur Theorie der algebraischen Potentialfunktionen des dreidimensionalen Raumes*, Math. Ann. 99 (1928), 629-659.
3. R. Gilbert, *Singularities of three-dimensional harmonic functions*, Pacific J. Math. 10 (1960), 1243-1255.
4. J. Hadamard, *Théorème sur les séries entières*, Acta Math. 22 (1898), 55-64.

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AN ALGORITHM OF J. SCHUR AND THE TAYLOR SERIES

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In a recent announcement in the Bulletin (cf. [1]), the authors outlined a proof that a differentiable function of a complex variable in a region R has a power series expansion in some neighborhood of each point of R (Theorem 3, [1]). The proof involved no integration, but failed to yield the full radius of convergence and required that it first be shown that all of the derivatives exist. In the last paragraph of [1], it was mentioned that a variation of an algorithm of J. Schur develops constructively the Taylor series for f , yields the full radius of convergence, and gives the existence of all the derivatives of f as a corollary. This procedure is developed in this paper.

The letter R will denote a bounded region in the complex plane.

LEMMA 1 (cf. [4, p. 77]). *If $f: \bar{R}$ into E_2 is continuous on $\bar{R}$, differentiable on R , and nonconstant, then*

$$|f(z)| < \max_{t \in \bar{R} - R} |f(t)| \quad \text{for all } z \in R.$$

LEMMA 2 (Theorem 1, [1]). *If $p \in R$ and f is differentiable and bounded on $R - p$, then f may be defined at p so that it will be continuous and differentiable at p .*

LEMMA 3. *If the polynomial, $P(z) = a_0 + a_1z + \cdots + a_nz^n$, satisfies $|P(z)| \leq 1$ for $|z| \leq 1$, then $|a_i| \leq 1$, $i = 0, 1, \cdots, n$.*

PROOF. The conclusion holds for all polynomials of degree zero. Suppose it holds for all polynomials of degree $\leq k$ and $P(z) = a_0$

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$+ a_{1z} + \dots + a_{k+1}z^{k+1}$ with $|P(z)| \leq 1$ for $|z| \leq 1$. If $Q(z) = 2^{-1}[P(z) - P(e^{i\theta}z)]$, $0 \leq \theta < 2\pi$, then $|Q(z)| \leq 1$ for $|z| \leq 1$. Now $Q(z)$ has no constant term and, consequently, $z^{-1}Q(z)$ is a polynomial of degree $\leq k$. Note that $|z^{-1}Q(z)| \leq 1$ for $|z| = 1$ and by Lemma 1 is ≤ 1 for $|z| \leq 1$. By the inductive hypothesis, the coefficients of $z^{-1}Q(z)$ are ≤ 1 in modulus. By letting $\theta = \pi n^{-1}$ we get $|a_n| \leq 1$ for $n = 1, 2, \dots, k+1$. Finally, $|a_0| = |P(0)| \leq 1$.

LEMMA 4. If $\sum_{p=0}^{\infty} a_p z^p$ converges for $|z| < 1$ and $|\sum_{p=0}^{\infty} a_p z^p| \leq 1$, for $|z| < 1$, then $|a_p| \leq 1$ for $p = 0, 1, \dots$.

PROOF. Let $0 < r < 1$ and $\epsilon > 0$. There exists $N > 0$ such that if $n > N$, $|\sum_{p=0}^n a_p r^p z^p| < 1 + \epsilon$ for $|z| \leq 1$. By Lemma 3, $|(1 + \epsilon)^{-1} \sum_{p=0}^n a_p r^p z^p| \leq 1$ for $p = 0, 1, \dots, n$. Thus $|a_p| \leq 1$ for all p .

LEMMA 5.¹ Let $f(z)$ be differentiable and $|f(z)| < 1$ for $|z| < 1$. Let $f_0(z) = f(z)$ and $f_{p+1}(z) = [e_p - f_p(z)](z[1 - \bar{e}_p f_p(z)])^{-1}$ where $e_p = f_p(0)$. Then either

- (1) $|e_p| < 1$ for all p and f_p will be defined and differentiable for $|z| < 1$ with $|f_p(z)| < 1$ for $|z| < 1$ and $p = 0, 1, \dots$, or
- (2) $|e_p| < 1$ for $p = 0, 1, \dots, (k-1)$ with $|e_k| = 1$ and f_p will be defined and differentiable for $|z| < 1$ and $p = 0, 1, \dots, k$ with $|f_p(z)| < 1$ for $|z| < 1$ and $p \leq k-1$ and $f_k(z) = \text{constant of modulus one}$.

PROOF. If h is a differentiable function with $|h(z)| < 1$ for $|z| < 1$ and $e = h(0)$, then $|[e - h(z)][1 - \bar{e}h(z)]^{-1}| < 1$ for $|z| < 1$ as is well known. Now, if $g(z) = [e - h(z)](z[1 - \bar{e}h(z)])^{-1}$, with $g(0)$ defined to be $-h'(0)(1 - |e|^2)^{-1}$, then g is continuous at $z = 0$ and by Lemma 2 is also differentiable there. Note that $|g(z)| \leq |z|^{-1}$ for $0 < |z| < 1$ and thus by Lemma 1, $|g(z)| \leq 1$ for $|z| < 1$. If $|g(z)| < 1$ for $|z| < 1$, then h may be replaced by g and the process continued. If there exists z_0 with $|z_0| < 1$ and $|g(z_0)| = 1$, then by Lemma 1, g is a constant of modulus one. This establishes Lemma 5.

LEMMA 6. Suppose that, under the hypothesis of Lemma 5, the first case holds and let, for $|z| < 1$ and $|t_p| \leq 1$ where t_p is arbitrarily chosen for $p = 0, 1, \dots$, $s_p(z, t_p) = [e_p - z t_p][1 - \bar{e}_p z t_p]^{-1} = e_p + (1 - |e_p|^2)z[\bar{e}_p z - t_p^{-1}]^{-1}$ and $S_p(z, t_p) = S_{p-1}(z, s_p(z, t_p))$, $p = 1, 2, \dots$, and $S_0(z, t_0) = s_0(z, t_0)$. Then $|S_p(z, t_p)| < 1$ and $S_p(z, t_{p+1}) = f_0(z)$ for $p = 0, 1, \dots$.

¹ Schur's technique is outlined on pages 285-288 of [3], where some basic identities from continued fraction theory and Schwarz's lemma are used. Hence Lemma 5 and Lemma 6 of this paper do no more than set up the continued fraction expansion for $f(z)$, and the proof of Lemma 7 avoids the use of the fundamental determinant formula for continued fractions.

PROOF. Since $|e_p| < 1$ and $|s_p(z, t_p)| < 1$, the first conclusion is immediate. For the second conclusion, note that $s_p(z, f_{p+1}(z)) = f_p(z)$ for $p = 0, 1, \dots$. Thus

$$\begin{aligned} S_p(z, f_{p+1}(z)) &= S_{p-1}(z, s_p[z, f_{p+1}(z)]) = S_{p-1}(z, f_p(z)) \cdots \\ &= S_0(z, f_1(z)) = s_0(z, f_1(z)) = f_0(z). \end{aligned}$$

LEMMA 7. *Under the hypotheses of Lemmas 5 and 6 we have, for $p = 0, 1, \dots$, that $S_p(z, t_p) = \sum_{m=0}^{\infty} a_m(p, t_p)z^m$ with the series converging to $S_p(z, t_p)$ for $|z| < 1$ and, moreover, the coefficients $a_m(p, t_p)$ are independent of p and t_p when $p \geq m$ and, finally, $|a_m(p, t_p)| \leq 1$ for all p and m .*

PROOF. The functions $S_p(z, t_p)$ are rational functions of z with a nonzero denominator provided $|z| < 1$ and, consequently have a power series expansion for $|z| < 1$. Now $a_0(p, t_p) = S_p(0, t_p) = f(0)$. Hence, suppose first that the coefficients are independent of t_p for $m \leq p$ and $p \leq n$. (Suppose, in fact, that this is true regardless of what function f is used originally.) By definition, $S_{n+1}(z, t_{n+1}) = S_n(z, s_{n+1}(z, t_{n+1})) = \cdots = s_0(z, s_1(z, s_2(z, \dots, s_{n+1}(z, t_{n+1})))) \cdots$ and this is $e_0 + (1 - |e|^2)z[e_0z - K^{-1}(z, t_{n+1})]^{-1}$ where $K(z, t_{n+1}) = s_1(z, s_2(z, s_3(z, \dots, s_{n+1}(z, t_{n+1})))) \cdots$. By the inductive hypothesis (with f replaced by f_1) the first n terms of the power series of $K(z, t_{n+1})$ are independent of t_{n+1} . Thus the first $n+1$ terms of the power series expansion of $e_0 + (1 - |e_0|^2)z[\bar{e}_0z - K^{-1}(z, t_{n+1})]$ are independent of t_{n+1} .

Now to show that $a_m(p, t_p)$ is independent of p for $p \geq m$, simply note that, since $S_{p+1}(z, t_{p+1}) = S_p(z, s_{p+1}(z, t_{p+1}))$, $a_m(p+1, t_{p+1}) = a_m(p, s_{p+1}(z, t_{p+1}))$ for all m and p . But for $p \geq m$, these are independent of t_{p+1} , so that $a_m(p+1) = a_m(p)$ for $p \geq m$. Finally, $|S_p(z, t_p)| < 1$ by Lemma 6 so that $|a_m(p, t_p)| \leq 1$ by Lemma 4.

This completes the proof of Lemma 7.

Let us now define, in view of Lemma 7,

$$a_m = a_m(p, t_p)$$

for $m = 0, 1, \dots, p$ and $p = 0, 1, \dots$.

THEOREM. *If f is differentiable for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$, then*

$$f(z) = \sum_{m=0}^{\infty} a_m z^m$$

for $|z| < 1$ and $|a_m| \leq 1$ for all m .

PROOF. Let us suppose that case 1 of Lemma 5 holds. The coefficients a_m are defined as above so that by Lemma 7, $|a_m| \leq 1$. Now when $p < q$,

$$\begin{aligned} |S_q(z, t_q) - S_p(z, t_p)| &= \left| \sum_{m=p+1}^{\infty} a_m(q, t_q) z^m - \sum_{m=p+1}^{\infty} a_m(p, t_p) z^m \right| \\ &\leq 2r^{p+1}(1-r)^{-1} \end{aligned}$$

whenever $|z| \leq r < 1$. Thus $\lim_p S_p(z, t_p)$ exists uniformly on compact subsets of $|z| < 1$ and is independent of the choice of t_p . By Lemma 6, $S_p(z, f_{p+1}(z)) = f_0(z) = f(z)$, thus $\lim_p S_p(z, t_p) = f(z)$. Now since

$$\begin{aligned} \left| \sum_{m=0}^{\infty} a_m z^m - \sum_{m=0}^{\infty} a_m(p, t_p) z^m \right| &= \left| \sum_{m=p+1}^{\infty} a_m z^m - \sum_{m=p+1}^{\infty} a_m(p, t_p) z^m \right| \\ &\leq 2r^{p+1}(1-r)^{-1}, \end{aligned}$$

it follows that $f(z) = \sum_{m=0}^{\infty} a_m z^m$.

Let us now suppose that case 2 of Lemma 5 holds so that $f_p(z)$ is defined only for $p=0, 1, \dots, k$ with $f_k(z) = e^{i\alpha}$. Also, s_p and S_p will be defined only for $p=0, 1, \dots, k$. Simply notice, as before, that $f(z) = S_{k-1}(z, f_k(z)) = S_{k-1}(z, e^{i\alpha}) = \sum_{m=0}^{\infty} a_m(k-1, e^{i\alpha}) z^m$. Thus, in this case, the coefficients are constants and a_m is defined to be $a_m(k-1, e^{i\alpha})$ and $f(z) = \sum_{m=0}^{\infty} a_m z^m$ as before.

COROLLARY. *If f is differentiable in a region R , then f' is continuous and differentiable in R .*

BIBLIOGRAPHY

1. P. Porcelli and E. H. Connell, *A proof of the power series expansion without Cauchy's formula*, Bull. Amer. Math. Soc. **67** (1961), 177-181.
2. J. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, Crelle's Journal **147** (1917), 205-232.
3. H. S. Wall, *Analytic theory of continued fractions*, Van Nostrand, New York, 1948.
4. G. T. Whyburn, *Topological analysis*, Princeton Univ. Press, Princeton, N. J., 1958.

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