ANALYTICITY WITHOUT ANALYSIS

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The purpose of this note is to point out an easy but striking consequence (Theorem 1) of a theorem proved by Walter Rudin in 1953, [1].

THEOREM 1. *Let X be a topological manifold of dimension two, and let f and g be functions sending X into the Riemann sphere. There exists a conformal structure for X relative to which both f and g are meromorphic and nonconstant if and only if every member of the complex algebra generated by f and g is continuous and interior and at least one member is also light, excepting constants.*

This "consequence" is striking because it enables one to define the various conformal structures carried by X using functions defined only on *all* of X and properties only of a topological or algebraic sort. Differentiability is never mentioned.

Since the algebra in question contains a member, h , which is light and interior, one concludes by Stoilow's theorem that h is locally topologically equivalent with power functions, i.e., h is a ramified cover of the sphere. Thus there exists a unique conformal structure for X relative to which h is meromorphic. For proof, see, e.g., [2, pp. 119–120]. If, relative to this structure, c is an analytic homeomorphism sending a disk into X , then the algebra generated by $h(c)$, $f(c)$, and $g(c)$ consists entirely of interior mappings (i.e., of mappings which send open sets onto open sets) and has one member, $h(c)$, which is known to be meromorphic. If these functions have no poles, then by Rudin's theorem (see Theorem 2 below) they are all analytic. Consequently, f and g are analytic (except for poles) on X relative to the same conformal structure. Riemann's theorem guarantees then that they are meromorphic on X relative to that structure. This completes the proof, for the converse is standard.

THEOREM 2 (RUDIN). *Let $\mathfrak{A}$ be a complex algebra of functions continuous on the open unit disk. Suppose there exists a basis, $\mathfrak{B}$, for that disk's topology, consisting entirely of Jordan domains such that $f|B$ attains its maximum modulus on the boundary of B whenever $f \in \mathfrak{A}$ and $B \in \mathfrak{B}$, provided B^- is in the open unit disk. If one nonconstant member of $\mathfrak{A}$ is analytic, then every member is analytic.*

Evidently, Theorem 2 may be applied whenever the relation

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$$(1) \quad \partial f(U) \subset f(\partial U)$$

holds for every $f \in \mathcal{A}$ and, e.g., every conditionally compact set U ; such functions are said by Whyburn [3, p. 112] to be quasi-open. Since a quasi-open and light map is necessarily interior, it follows that one may replace "interior" in Theorem 1 with "quasi-open." It is important to note how much weaker than quasi-openness are Rudin's hypotheses in Theorem 2 so far as individual functions are concerned. For example, Rudin's hypotheses (applied to a *single* function) permit *folding* whereas quasi-openness does not, as one sees easily upon consideration of the map f defined by the rules

$$(2) \quad f(z) = \begin{cases} z & \text{when } |z| \leq 1, \\ \frac{z}{|z|^2} & \text{when } |z| \geq 1. \end{cases}$$

Indeed, Whyburn [4] has shown recently that those mappings whose restrictions to multiply connected conditionally compact Jordan domains either have zero minimum modulus (computed relative to an *arbitrary* point as origin) or else attain their minimum modulus on the boundary are exactly the quasi-open mappings. Thus Rudin's extreme modulus property (for a *single* function) is required to hold only relative to the origin whereas Whyburn's must hold relative to *every* point serving as the origin.

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