A NOTE ON THE BOUNDEDNESS OF SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

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Hartman and Wintner [1] obtained a Sturmian comparison theorem for self-adjoint second order elliptic equations of the form

$$(1) \quad \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + fu = 0, \quad a_{ij} = a_{ji},$$

in a bounded domain B with boundary ∂B . In this note, their method is slightly modified to prove the following theorem.

Denote by D the semi-infinite cylinder $\{(x, t) : x \in B, t > 0\}$, by $\bar{D}$ its closure and by D_T the intersection of D with the half-space $t \leq T$. Suppose $u = u(x)$ is a solution of equation (1) which is continuous in $\bar{B}$ the closure of B , vanishes on ∂B and has continuous second derivatives in B . Again, suppose $w = w(x, t)$ is defined and continuous in the closed region $\bar{D}_T$, is positive on B at $t = 0$ and on ∂B for all $t \geq 0$ and has continuous derivatives $\partial w / \partial t$, $\partial^2 w / \partial x_i \partial x_j$ which satisfy the parabolic equation

$$(2) \quad \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial w}{\partial x_j} \right) + Fw = C \frac{\partial w}{\partial t}, \quad A_{ij} = A_{ji},$$

in D_T for all $T > 0$. The functions A_{ij} , $\partial A_{ij} / \partial x_i$, F and C are uniformly bounded continuous functions of x_i and t in D_T for any given $T > 0$, while C is bounded in D by two positive constants C_0 and C_1 ($0 < C_0 \leq C \leq C_1$) and the quadratic form $\sum_{i,j} A_{ij} \xi_i \xi_j$ is non-negative at all points in D .

THEOREM 1. *If $\int_B \{u^2(F-f) + \sum_{i,j=1}^n (a_{ij} - A_{ij})(\partial u / \partial x_i)(\partial u / \partial x_j)\} d\tau \geq \epsilon(t)$ for all $t > 0$ and $\int_0^T \epsilon(t) dt$ tends to infinity with T , then w is unbounded in D .*

PROOF. Since $w > 0$ on $\bar{B}$ at $t = 0$ and on ∂B for all $t > 0$, the maximum principle for parabolic equations (see [2]) implies $w > 0$ in D_T for all $T > 0$.

The Green identity leads from the boundary condition $u = 0$ on ∂B , to the divergence relation

$$(3) \quad \int_B \left\{ \sum_{i=1}^n \frac{\partial}{\partial x_i} (u^2 h^i) \right\} d\tau = 0 \quad \text{for all } t \geq 0,$$

Received by the editors November 14, 1960 and, in revised form, February 20, 1961.

where h^i is the t -dependent vector function

$$(4) \quad h^i = \sum_{j=1}^n \frac{A_{ij}}{w} \frac{\partial w}{\partial x_j}.$$

This vector is finite at each point of D since $w > 0$ in this region.

Since u is a solution of (1) and vanishes on ∂B , another application of Green's identity shows that

$$(5) \quad \int_B \left\{ \sum_{i,j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - f u^2 \right\} d\tau = 0.$$

It is clear from (4) and (2) that

$$(6) \quad \sum_{i=1}^n \frac{\partial h^i}{\partial x_i} = \frac{C}{w} \frac{\partial w}{\partial t} - F - \sum_{i,j=1}^n \frac{A_{ij}}{w^2} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j}$$

and using this in conjunction with (3) and (5), we see that

$$(7) \quad \int_B \frac{u^2 C}{w} \frac{\partial w}{\partial t} d\tau = \int_B \left\{ \sum_{i,j=1}^n \left(a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - 2 \frac{u}{w} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} + \frac{u^2}{w^2} A_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \right) + u^2 (F - f) \right\} d\tau,$$

for all $t \geq 0$. The right-hand side of this equation can be rewritten in the form

$$(8) \quad \int_B \left\{ \sum_{i,j=1}^n A_{ij} \left(\frac{\partial u}{\partial x_i} - \frac{u}{w} \frac{\partial w}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} - \frac{u}{w} \frac{\partial w}{\partial x_j} \right) \right\} d\tau + \int_B \left\{ u^2 (F - f) + \sum_{i,j=1}^n (a_{ij} - A_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right\} d\tau,$$

since $A_{ij} = A_{ji}$.

Since the quadratic form $\sum_{i,j=1}^n A_{ij} \xi_i \xi_j$ is positive definite or at least non-negative and the second term of expression (8) is assumed greater than or equal to $\epsilon(t)$, we see from (7) that $\partial z / \partial t \geq \epsilon(t)$, where $z = \int_B C u^2 \log w d\tau$. But then $z(T) \geq z(0) + \int_0^T \epsilon(t) dt$ and is unbounded if $\int_0^T \epsilon(t) dt$ tends to infinity with T . If z is unbounded, w cannot be bounded; for if it were bounded by w_0 for all T we would have $z(T) \leq \int_B C_1 u^2 \log w_0 d\tau$ always.

Protter [3] extended the comparison theorem of Hartman and Wintner to a form valid for a pair of general linear elliptic equations. A corresponding extension of Theorem 1 is given below.

THEOREM 2. *If u and w satisfy the general linear equations*

$$(9) \quad \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + fu = 0,$$

$$(10) \quad \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial w}{\partial x_j} \right) + \sum_{i=1}^n B_i \frac{\partial w}{\partial x_i} + Fw = C \frac{\partial w}{\partial t},$$

in B and D , and the boundary and continuity conditions of Theorem 1, then w is unbounded in D if $\int_0^T \epsilon(t) dt$ tends to infinity with T and

$$(11) \quad \int_B \left\{ \sum_{i,j=1}^n (a_{ij} - A_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{u^2}{4} \left[4(F - f) - \sum_{i,j=1}^n A_{ij} B_i B_j + 2 \sum_{i=1}^n \left(\frac{\partial b_i}{\partial x_i} - \frac{\partial B_i}{\partial x_i} \right) \right] \right\} d\tau$$

is greater than or equal to $\epsilon(t)$ for all $t > 0$. A^{ij} denote the elements of the inverse matrix of A_{ij} and the functions $\partial b_i / \partial x_i$, $\partial B_i / \partial x_i$ are assumed continuous in $\bar{B}$ and $\bar{D}$ respectively.

PROOF. Let us construct an integral corresponding to (7) with its left-hand side, like (8), consisting of an integral of a non-negative form containing all terms involving the function w , and a residual integral containing only terms involving the function u and the coefficients of equations (9) and (10).

Consider the expression

$$(12) \quad E = \int_B \left\{ \sum_{i,j=1}^n (a_{ij} - A_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^n u \frac{\partial u}{\partial x_i} (B_i - b_i) + (F - f)u^2 \right\} d\tau + \int_B \left\{ \sum_{i,j=1}^n A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \sum_{i=1}^n u \frac{\partial u}{\partial x_i} B_i - Fu^2 \right\} d\tau.$$

The term Fu^2 of the second integral can be eliminated by subtracting from it the vanishing integral

$$(13) \quad E' = \int_B \left\{ \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(u^2 \sum_{j=1}^n \frac{A_{ij}}{w} \frac{\partial w}{\partial x_j} \right) \right\} d\tau = \int_B \left\{ \frac{Cu^2}{w} \frac{\partial w}{\partial t} - Fu^2 - \frac{u^2}{w} \sum_{i=1}^n B_i \frac{\partial w}{\partial x_i} - \sum_{i,j=1}^n \left(\frac{u^2}{w^2} A_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} - 2 \frac{u}{w} A_{ij} \frac{\partial u}{\partial x_i} \frac{\partial w}{\partial x_j} \right) \right\} d\tau.$$

Since E is zero by virtue of (9),

$$\begin{aligned}
 \frac{\partial z}{\partial t} &= \int_B \frac{Cu^2}{w} \frac{\partial w}{\partial t} d\tau \\
 &= \int_B \left\{ \sum_{i,j=1}^n (a_{ij} - A_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right. \\
 &\quad \left. + \sum_{i=1}^n u \frac{\partial u}{\partial x_i} (B_i - b_i) + (F - f)u^2 \right\} d\tau \\
 &\quad + \int_B \left\{ \sum_{i,j=1}^n A_{ij} \left(\frac{\partial u}{\partial x_i} - \frac{u}{w} \frac{\partial w}{\partial x_i} \right) \left(\frac{\partial u}{\partial x_j} - \frac{u}{w} \frac{\partial w}{\partial x_j} \right) \right. \\
 (14) \quad &\quad \left. - \sum_{i=1}^n u B_i \left(\frac{\partial u}{\partial x_i} - \frac{u}{w} \frac{\partial w}{\partial x_i} \right) \right\} d\tau \\
 &= \int_B \left\{ \sum_{i,j=1}^n (a_{ij} - A_{ij}) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right. \\
 &\quad \left. + \frac{u^2}{4} \left[4(F - f) - A^{ij} B_i B_j + 2 \sum_{i=1}^n \left(\frac{\partial b_i}{\partial x_i} - \frac{\partial B_i}{\partial x_i} \right) \right] \right\} d\tau \\
 &\quad + \int_B \left\{ \sum_{i,j=1}^n A_{ij} \left(\frac{\partial u}{\partial x_i} - \frac{u}{w} \frac{\partial w}{\partial x_i} - \frac{u}{2} \sum_{k=1}^n A^{ik} B_k \right) \right. \\
 &\quad \left. \cdot \left(\frac{\partial u}{\partial x_j} - \frac{u}{w} \frac{\partial w}{\partial x_j} - \frac{u}{2} \sum_{k=1}^n A^{jk} B_k \right) \right\} d\tau
 \end{aligned}$$

where A^{ij} is the inverse matrix of A_{ij} . The matrix A_{ij} is required to be nonsingular in Theorem 2 as well as non-negative in D . Since $\sum_{i,j=1}^n A_{ij} \xi_i \xi_j$ is positive definite in D the second integral of (14) is non-negative. The remainder of the proof now proceeds as for Theorem 1.

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