ON THE ABSOLUTE CESÀRO SUMMABILITY OF FOURIER SERIES

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1. Let $f(t)$ be an even integrable function with period 2π and let

$$f(t) \sim \sum_{n=1}^{\infty} a_n \cos nt,$$

$$\Phi_{\alpha}(t) = \int_0^t (t-u)^{\alpha-1} f(u) du, \quad \alpha > 0.$$

L. S. Bosanquet [2] proved the following:

THEOREM A. *If $\Phi_{\alpha}(t)/t^{\alpha}$ is of bounded variation in $(0, \pi)$, then, for $\gamma > \alpha$, the Fourier series of $f(t)$ at $t=0$ is evaluable $|C, \gamma|$, and conversely, if the Fourier series of $f(t)$ at $t=0$ is evaluable $|C, \gamma|$, then, for $\alpha > \gamma + 1$, $\Phi_{\alpha}(t)/t^{\alpha}$ is of bounded variation in $(0, \pi)$.*

Earlier, L. S. Bosanquet [1] proved the following:

THEOREM B. *If $\Phi_{\alpha}(t)/t^{\alpha}$ is of bounded variation in an interval to the right of $t=0$, then the Fourier series of $f(t)$ at $t=0$ is evaluable $(C, \alpha-1)$ when $\alpha \geq 1$.*

The purpose of this paper is to establish the following:

THEOREM. *If the Fourier series of $f(t)$ at $t=0$ is evaluable $|C, \alpha|$ to zero, then $\Phi_{\alpha}(t) = o(t^{\alpha})$ when $\alpha \geq 1$.*

This theorem was implicitly proved by N. Obreschkoff [4, Satz 2] when $\alpha = 1$.

2. Preliminary lemmas.

LEMMA 1. *Let a series $\sum_{n=1}^{\infty} b_n$ be convergent and let*

$$c_n = \sum_{r=n}^{\infty} b_r.$$

Then

$$(2.1) \quad \sum_{r=n}^m a_r b_r = a_n c_n - a_m c_{m+1} - \sum_{r=n}^{m-1} c_{r+1} \Delta a_r,$$

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where $\Delta a_r = a_r - a_{r+1}$. In particular, if $a_m c_{m+1} \rightarrow 0$ as $m \rightarrow \infty$,

$$(2.2) \quad \sum_{r=n}^{\infty} a_r b_r = a_n c_n - \sum_{r=n}^{\infty} c_{r+1} \Delta a_r.$$

Proof is obvious.

Throughout this paper, let

$[x]$ be the greatest integer less than x ,

$$\beta = [\alpha],$$

$$A_n^\gamma = \binom{\gamma + n}{n} = \frac{\Gamma(n + \gamma + 1)}{\Gamma(n + 1)\Gamma(\gamma + 1)},$$

$$s_n^\gamma = \sum_{r=0}^n A_{n-r}^\gamma a_r,$$

$$\phi(n, t) = \int_0^t (t^2 - u^2)^{\alpha-1} \cos nu \, du,$$

$$\Delta^k \phi(n, t) = \Delta^{k-1} \phi(n, t) - \Delta^{k-1} \phi(n+1, t), \quad \Delta^0 \phi(n, t) = \phi(n, t),$$

$$\psi(\mu, t) = 2^\mu \Gamma(\mu + 1) t^{-\mu} J_\mu(t),$$

where $J_\mu(t)$ is the Bessel function of order μ , $\mu > -1/2$.

LEMMA 2. For $\alpha > 0$ and $k = 0, 1, 2, \dots$,

$$(2.3) \quad \Delta^k \phi(n, t) = O(n^{-\alpha} t^{\alpha+k-1}).$$

PROOF. Since

$$(2.4) \quad \psi\left(\alpha - \frac{1}{2}, nt\right) = \frac{2\Gamma(\alpha + 1/2)}{\Gamma(1/2)\Gamma(\alpha)} t^{1-2\alpha} \phi(n, t),$$

we have the lemma by a known result. (See K. Chandrasekharan and O. Szász [3, pp. 716 and 728].)

LEMMA 3. For $\alpha > 0$ and $j = 0, 1, 2, \dots$, such that $\alpha + j \geq 1$,

$$(2.5) \quad \sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+j} \phi(n, t) = O(\nu^{-\alpha} t^{2\alpha-2+j}).$$

PROOF. The lemma in which α is an integer is obvious by (2.3), so that we shall prove the lemma in which α is not an integer. Putting $\rho = [t^{-1}]$ and using (2.2) and (2.3),

$$\begin{aligned}
\sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+j} \phi(n, t) &= \left(\sum_{n=\nu}^{\nu+\rho-1} + \sum_{n=\nu+\rho}^{\infty} \right) \\
&= O \left(t^{\alpha+\beta+j-1} \sum_{n=\nu}^{\nu+\rho-1} A_{n-\nu}^{\beta-\alpha} n^{-\alpha} \right) + A_{\rho}^{\beta-\alpha} \Delta^{\beta+j-1} \phi(\nu + \rho, t) \\
&\quad + \sum_{n=\nu+\rho}^{\infty} A_{n-\nu+1}^{\beta-\alpha-1} \Delta^{\beta+j-1} \phi(n+1, t) \\
&= O(\nu^{-\alpha} t^{\alpha+\beta+j-1} A_{\rho-1}^{\beta-\alpha+1}) + O(\nu^{-\alpha} t^{\alpha+\beta+j-2} \cdot \rho^{\beta-\alpha}) \\
&\quad + O \left(t^{\alpha+\beta+j-2} \sum_{n=\nu+\rho}^{\infty} (n - \nu + 1)^{\beta-\alpha-1} n^{-\alpha} \right) \\
&= O(\nu^{-\alpha} t^{2\alpha+j-2}) + O(\nu^{-\alpha} t^{\alpha+\beta+j-2} \cdot \rho^{\beta-\alpha}) = O(\nu^{-\alpha} t^{2\alpha-2+j}),
\end{aligned}$$

which is the required result.

LEMMA 4. Let $\alpha \geq 1$. Then, for any m and $t > 0$,

$$(2.6) \quad U_m(t) \equiv \sum_{\nu=1}^m A_{\nu}^{\alpha} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+2} \phi(n, t) = O(t^{2\alpha-1}).$$

PROOF. If $mt \leq 1$, then, using (2.5) for $j=2$,

$$(2.7) \quad U_m(t) = O \left(\sum_{\nu=1}^m A_{\nu}^{\alpha} \cdot \nu^{-\alpha} \cdot t^{2\alpha} \right) = O(mt^{2\alpha}) = O((mt) \cdot t^{2\alpha-1}) = O(t^{2\alpha-1}).$$

Therefore we shall prove (2.6) when $mt > 1$. Putting $\rho = [t^{-1}]$, we write

$$U_m(t) = \left(\sum_{\nu=1}^{\rho-1} + \sum_{\nu=\rho}^m \right) = U_{m1}(t) + U_{m2}(t),$$

where $U_{m1}(t) = O(t^{2\alpha-1})$ by (2.7). Since, for $j=0, -1$, using (2.5),

$$\begin{aligned}
&\sum_{\nu=k}^{\infty} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+2+j} \phi(n, t) \\
&= \lim_{p \rightarrow \infty} \sum_{\nu=k}^p \left(\sum_{n=0}^{\infty} A_n^{\beta-\alpha} \Delta^{\beta+1+j} \phi(n + \nu, t) - \sum_{n=0}^{\infty} A_n^{\beta-\alpha} \Delta^{\beta+1+j} \phi(n + \nu + 1, t) \right) \\
&= \lim_{p \rightarrow \infty} \left(\sum_{n=0}^{\infty} A_n^{\beta-\alpha} \Delta^{\beta+1+j} \phi(n + k, t) - \sum_{n=0}^{\infty} A_n^{\beta-\alpha} \Delta^{\beta+1+j} \phi(n + p + 1, t) \right) \\
&= \sum_{n=k}^{\infty} A_{n-k}^{\beta-\alpha} \Delta^{\beta+1+j} \phi(n, t),
\end{aligned}$$

we have, by (2.1) and (2.5),

$$\begin{aligned}
 U_{m2}(t) &= A_\rho^\alpha \sum_{n=\rho}^{\infty} A_{n-\rho}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t) - A_m^\alpha \sum_{n=m+1}^{\infty} A_{n-m-1}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t) \\
 &\quad + \sum_{\nu=\rho}^{m-1} A_{\nu+1}^{\alpha-1} \sum_{n=\nu+1}^{\infty} A_{n-\nu-1}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t) \\
 &= A_\rho^\alpha \sum_{n=\rho}^{\infty} A_{n-\rho}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t) - A_m^\alpha \sum_{n=m+1}^{\infty} A_{n-m-1}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t) \\
 &\quad + A_{\rho+1}^{\alpha-1} \sum_{n=\rho+1}^{\infty} A_{n-\rho-1}^{\beta-\alpha} \Delta^\beta \phi(n, t) - A_m^{\alpha-1} \sum_{n=m+1}^{\infty} A_{n-m-1}^{\beta-\alpha} \Delta^\beta \phi(n, t) \\
 &\quad + \sum_{\nu=\rho}^{m-2} A_{\nu+2}^{\alpha-2} \sum_{n=\nu+2}^{\infty} A_{n-\nu-2}^{\beta-\alpha} \Delta^\beta \phi(n, t) \\
 &= O(\rho^\alpha \cdot \rho^{-\alpha} \cdot t^{2\alpha-1}) + O(m^\alpha \cdot m^{-\alpha} \cdot t^{2\alpha-1}) + O(\rho^{\alpha-1} \cdot \rho^{-\alpha} \cdot t^{2\alpha-2}) \\
 &\quad + O(m^{\alpha-1} \cdot m^{-\alpha} t^{2\alpha-2}) + O\left(t^{2\alpha-2} \sum_{\nu=\rho}^{\infty} \nu^{\alpha-2} \cdot \nu^{-\alpha}\right) \\
 &= O(t^{2\alpha-1}) + O(m^{-1} \cdot t^{2\alpha-2}) + O(\rho^{-1} \cdot t^{2\alpha-2}) \\
 &= O(t^{2\alpha-1}),
 \end{aligned}$$

which is the required result.

LEMMA 5. If the series $\sum_{n=0}^{\infty} a_n$ with $a_0=0$ is evaluable $|C, \alpha|$, $\alpha > 0$, to zero, then the series $\sum_{n=1}^{\infty} a_n \phi(n, t)$ is convergent for any $t > 0$ and

$$\sum_{n=1}^{\infty} a_n \phi(n, t) = \sum_{\nu=1}^{\infty} s_\nu^{\alpha-1} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t).$$

REMARK. It is remarkable that in this lemma the series $\sum_{n=1}^{\infty} a_n \cos nt$ need not be a Fourier series.

PROOF. The summability $|C, \alpha|$ of the series $\sum_{n=0}^{\infty} a_n$ implies that the series $\sum_{n=1}^{\infty} n^{-\alpha} |a_n|$ is convergent, by the well-known theorem. Hence, by (2.3), it follows that the series $\sum_{n=1}^{\infty} a_n \phi(n, t)$ is convergent for any $t > 0$. By the repeated use of Abel's transformation,

$$\sum_{n=1}^m a_n \phi(n, t) = \sum_{n=1}^{m-\beta-1} s_n^\beta \Delta^{\beta+1} \phi(n, t) + \sum_{j=0}^{\beta} s_{m-j}^j \Delta^j \phi(m-j, t),$$

where, by $s_n^j = o(n^\alpha)$ when $0 \leq j \leq \alpha$ and (2.3),

$$\sum_{j=0}^{\beta} s_{m-j}^j \Delta^j \phi(m-j, t) = o(1) \quad \text{as } m \rightarrow \infty.$$

Thus

$$\sum_{n=1}^{\infty} a_n \phi(n, t) = \sum_{n=1}^{\infty} s_n^{\beta} \Delta^{\beta+1} \phi(n, t).$$

By the well-known formula

$$s_n^{\beta} = \sum_{r=0}^n A_{n-r}^{\beta-\alpha} s_r^{\alpha-1} = \sum_{r=1}^n A_{n-r}^{\beta-\alpha} s_r^{\alpha-1},$$

we have formally

$$\begin{aligned} \sum_{n=1}^{\infty} s_n^{\beta} \Delta^{\beta+1} \phi(n, t) &= \sum_{n=1}^{\infty} \Delta^{\beta+1} \phi(n, t) \sum_{r=1}^n A_{n-r}^{\beta-\alpha} s_r^{\alpha-1} \\ &= \sum_{r=1}^{\infty} s_r^{\alpha-1} \sum_{n=r}^{\infty} A_{n-r}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t). \end{aligned}$$

Thus, if we may prove that this interchange is legitimate, then the proof is complete. For this purpose, it is sufficient to prove that, for a fixed $t > 0$,

$$(2.8) \quad \lim_{N \rightarrow \infty} \sum_{r=1}^N s_r^{\alpha-1} \sum_{n=N+1}^{\infty} A_{n-r}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t) = 0.$$

Since, by (2.3),

$$(2.9) \quad \sum_{r=n}^{\infty} \Delta^{\beta+1} \phi(r, t) = \Delta^{\beta} \phi(n, t),$$

we have (2.8) for an integer α , using $s_n^{\alpha} = o(n^{\alpha})$ and (2.3). Hence we shall prove (2.8) in which α is not an integer. By (2.2), (2.3) and (2.9), for ν , $1 \leq \nu \leq N$,

$$\begin{aligned} &\sum_{n=N+1}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t) \\ (2.10) \quad &= A_{N-\nu+1}^{\beta-\alpha} \Delta^{\beta} \phi(N+1, t) + \sum_{n=N+1}^{\infty} A_{n-\nu+1}^{\beta-\alpha-1} \Delta^{\beta} \phi(n+1, t) \\ &= O((N-\nu+1)^{\beta-\alpha} N^{-\alpha}). \end{aligned}$$

Now the summability $|C, \alpha|$ of the series $\sum_{n=0}^{\infty} a_n$ implies

$$(2.11) \quad \sum_{r=1}^n |s_r^{\alpha-1}| = O(n^{\alpha}),$$

by the series analogue of Rajagopal's Lemma [5, Lemma 10]. Then, let us write, putting $\eta = [N/2]$,

$$\sum_{\nu=1}^N s_{\nu}^{\alpha-1} \sum_{n=N+1}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+1} \phi(n, t) = \left(\sum_{\nu=1}^{\eta-1} + \sum_{\nu=\eta}^N \right) = I_1 + I_2,$$

where, by (2.10) and (2.11),

$$\begin{aligned} I_1 &= O \left(\sum_{\nu=1}^{\eta-1} |s_{\nu}^{\alpha-1}| \cdot (N - \nu + 1)^{\beta-\alpha} \cdot N^{-\alpha} \right) \\ &= O \left(N^{\beta-2\alpha} \sum_{\nu=1}^{\eta-1} |s_{\nu}^{\alpha-1}| \right) = O(N^{\beta-\alpha}) = o(1). \end{aligned}$$

For I_2 , we take an arbitrarily fixed number L , but large enough. Then, using (2.10), (2.11) and $s_n^{\alpha-1} = o(n^{\alpha})$,

$$\begin{aligned} I_2 &= O \left(N^{-\alpha} \sum_{\nu=\eta}^N |s_{\nu}^{\alpha-1}| \cdot (N - \nu + 1)^{\beta-\alpha} \right) \\ &= O \left(N^{-\alpha} \sum_{\nu=1}^{N-\eta+1} |s_{N-\nu+1}^{\alpha-1}| \cdot \nu^{\beta-\alpha} \right) \\ &= O \left(N^{-\alpha} \sum_{\nu=1}^{L-1} |s_{N-\nu+1}^{\alpha-1}| \cdot \nu^{\beta-\alpha} \right) + O \left(N^{-\alpha} \sum_{\nu=L}^{N-\eta+1} |s_{N-\nu+1}^{\alpha-1}| \cdot \nu^{\beta-\alpha} \right) \\ &= O \left(\sum_{\nu=1}^L \nu^{\beta-\alpha} \right) + O \left(N^{-\alpha} \sum_{\nu=1}^L |s_{N-\nu+1}^{\alpha-1}| \cdot L^{\beta-\alpha} \right) \\ &\quad + O \left(N^{-\alpha} \sum_{\nu=1}^{N-\eta+1} |s_{N-\nu+1}^{\alpha-1}| \cdot (N - \eta + 1)^{\beta-\alpha} \right) \\ &\quad + O \left(N^{-\alpha} \sum_{\nu=L}^{N-\eta} \left(\sum_{n=1}^{\nu} |s_{N-n+1}^{\alpha-1}| \right) \cdot \nu^{\beta-\alpha-1} \right) \\ &= o(1) + O(L^{\beta-\alpha}) + O(N^{\beta-\alpha}) + O(L^{\beta-\alpha}) \\ &= o(1) + O(L^{\beta-\alpha}). \end{aligned}$$

Since L is arbitrary, we have $I_2 = o(1)$ by $\beta - \alpha < 0$. Thus (2.8) in which α is not an integer is valid.

3. Proof of theorem. The notations in the former paragraphs are used without further explanations. We shall now prove the theorem in which α is not an integer, the remaining part¹ being obtained by the analogous method. In this case, by the Chandrasekharan and Szász Theorem [3, Theorem 5] for the proof, it is sufficient to prove that

¹ This part is obtained as a corollary of Obreschkoff's Theorem [4, Satz 2] using the following Chandrasekharan and Szász Theorem. This remark is due to Professor G. Sunouchi.

$$\Phi_{\alpha}^{*}(t) \equiv \int_0^t (t^2 - u^2)^{\alpha-1} f(u) du = o(t^{2\alpha-1}).$$

Since $\alpha \geq 1$, we have

$$\Phi_{\alpha}^{*}(t) = \sum_{n=1}^{\infty} a_n \phi(n, t).$$

By Lemma 5 and the Abel transformation, using (2.5), (2.6) and $s_n^{\alpha} = o(n^{\alpha})$,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \phi(n, t) &= \sum_{\nu=1}^{\infty} s_{\nu}^{\alpha} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+2} \phi(n, t) \\ &= \sum_{\nu=1}^{\infty} \sigma_{\nu}^{\alpha} \left(A_{\nu}^{\alpha} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+2} \phi(n, t) \right) \\ &= \sum_{\nu=1}^{\infty} (\sigma_{\nu}^{\alpha} - \sigma_{\nu+1}^{\alpha}) U_{\nu}(t), \end{aligned}$$

where $\sigma_{\nu}^{\alpha} = s_{\nu}^{\alpha} / A_{\nu}^{\alpha}$. Since the series is evaluable $|C, \alpha|$, that is,

$$\sum_{\nu=1}^{\infty} |\sigma_{\nu}^{\alpha} - \sigma_{\nu+1}^{\alpha}| < +\infty,$$

using (2.6), for an arbitrary positive number ϵ , there exists an $N = N(\epsilon)$ such that

$$\left| \sum_{\nu=N}^{\infty} (\sigma_{\nu}^{\alpha} - \sigma_{\nu+1}^{\alpha}) U_{\nu}(t) \right| = O \left(t^{2\alpha-1} \sum_{\nu=N}^{\infty} |\sigma_{\nu}^{\alpha} - \sigma_{\nu+1}^{\alpha}| \right) < \epsilon t^{2\alpha-1}.$$

On the other hand, for a fixed $\nu > 0$, by (2.3),

$$\begin{aligned} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+2} \phi(n, t) &= O \left(t^{\alpha+\beta+1} \sum_{n=\nu}^{\infty} (n - \nu + 1)^{\beta-\alpha} n^{-\alpha} \right) \\ &= O(t^{\alpha+\beta+1}). \end{aligned}$$

Then, for $m \leq N$,

$$U_m(t) = \sum_{\nu=1}^m A_{\nu}^{\alpha} \sum_{n=\nu}^{\infty} A_{n-\nu}^{\beta-\alpha} \Delta^{\beta+2} \phi(n, t) = O \left(t^{\alpha+\beta+1} \sum_{\nu=1}^m A_{\nu}^{\alpha} \right) = O(t^{\alpha+\beta+1} N^{\alpha+1}).$$

Hence

$$\lim_{t \rightarrow 0+} \sum_{\nu=1}^N (\sigma_{\nu}^{\alpha} - \sigma_{\nu+1}^{\alpha}) U_{\nu}(t) / t^{2\alpha-1} = 0.$$

Therefore we have

$$\limsup_{t \rightarrow 0+} \left| \sum_{\nu=1}^{\infty} (\sigma_{\nu}^{\alpha} - \sigma_{\nu+1}^{\alpha}) U_{\nu}(t) / t^{2\alpha-1} \right| \leq \epsilon.$$

Since ϵ is arbitrary, we get

$$\lim_{t \rightarrow 0+} \sum_{\nu=1}^{\infty} (\sigma_{\nu}^{\alpha} - \sigma_{\nu+1}^{\alpha}) U_{\nu}(t) / t^{2\alpha-1} = 0,$$

that is,

$$\lim_{t \rightarrow 0+} \Phi_{\alpha}^{*}(t) / t^{2\alpha-1} = 0,$$

and the theorem in which α is not an integer is completely proved.

4. Bessel summability. A series $\sum_{n=1}^{\infty} c_n$ is said to be evaluable (J_{μ}) to zero if the series

$$\sum_{n=1}^{\infty} c_n \psi(\mu, nt)$$

converges in some interval $0 < t < t_0$, and if its sum tends to zero as $t \rightarrow 0$. From the argument in §3, we have, by (2.4),

COROLLARY. A series evaluable $|C, \alpha|$ to zero is also evaluable $(J_{\alpha-1/2})$ to zero when $\alpha \geq 1$.

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