$Rax_1 + Rax_2 + \cdots + Rax_n, \cdots$. It must terminate; thus for some n , $Rax_i \subset Rax_1 + \cdots + Rax_n$ for all ax_i such that $L(ax_i)$ is maximal. Now ax_1r is either 0 or a maximal ax_i ; whence $Rax_1R \subset Rax_1 + \cdots + Rax_n$. But there is a $u \neq 0$ in Rax_1R such that $ax_1u = \cdots = ax_nu = 0$. Thus $Rax_1Ru \subset Rax_1u + \cdots + Rax_nu = (0)$. Therefore, since $Ru \neq 0 \subset Rax_1R$, $(Ru)^2 \subset Rax_1RRu = (0)$. We have produced a nonzero nilpotent left-ideal in R ! This contradiction proves the theorem.

Clearly the proof would work if A were a nil right-ideal, for if $0 \neq a \in A$, then Ra would be a nonzero nil left-ideal of R .

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A REMARK ON DIRECT PRODUCTS OF MODULES

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It is a standard result of abelian group theory that the direct product of an infinite number of infinite cyclic groups is not free [3, p. 48]. Various generalizations of this fact to modules over an arbitrary ring are contained in [1]. In this note we restrict our attention to integral domains, and in this setting present a generalization of the aforementioned result which is similar to, but in some sense stronger than, those of [1]. We then apply our theorem to show that any factor group A of a direct product of infinite cyclic groups possesses the following amusing property: If A is a subgroup of a direct sum of reduced torsion-free abelian groups, then it must be contained in the direct sum of a finite number of them.

Throughout this discussion, R will be an integral domain which is not a field. All R -modules considered will be assumed to be unitary.

LEMMA. *Let A be a torsion-free R -module, and set $A' = \bigcap_{\lambda \in R^*} \lambda A$. Then A' is divisible. In particular, if A is reduced (i.e., has no divisible submodules) then $\bigcap_{\lambda \in R^*} \lambda A = 0$.¹*

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¹ R^* means $R - \{0\}$.

PROOF. Let $x \in A'$ and $\lambda \in R^*$; then $x \in \lambda A$, and so there exists $y \in A$ such that $x = \lambda y$. But then, if $\mu \in R^*$, $x \in \lambda \mu A$, and so, since A is torsion-free, $y \in \mu A$. This is true for all $\mu \in R^*$, and so $y \in A'$. It then follows immediately that A' is divisible.

THEOREM. Let $\{A^{(i)}\}$ be a sequence of nontrivial R -modules, and set $A_n = \prod_{i=n+1}^{\infty} A^{(i)}$, $A = A_0 = \prod_{i=1}^{\infty} A^{(i)}$ (then, of course, A_n is a direct summand of A). Let $\{C_\alpha\}$ be a family of torsion-free reduced R -modules, where α traces some index set I , and set $C = \sum_\alpha C_\alpha$. Let $f: A \rightarrow C$ be an R -homomorphism, and let $f_\alpha: A \rightarrow C_\alpha$ be the composition of f with the projection of C onto C_α . Then there exists $n \geq 0$ such that $f_\alpha(A_n) = 0$ for all but finitely many $\alpha \in I$.

PROOF. Assume the statement is false; then, for any $n \geq 0$, $f_\alpha(A_n) \neq 0$ for infinitely many $\alpha \in I$. We shall construct inductively sequences $\{\alpha_n\} \subseteq I$, $\{x_n\} \subseteq A$, $\{\lambda_n\} \subseteq R^*$ such that the following conditions are satisfied:

- (a) $x_n \in A_n$.
- (b) $f_{\alpha_n}(x_k) = 0$ for $k < n$.
- (c) $f_{\alpha_n}(x_n) \neq 0 \pmod{\lambda_n C_{\alpha_n}}$.

Select $\alpha_1 \in I$ such that $f_{\alpha_1}(A_1) \neq 0$; then there exists $x_1 \in A_1$ such that $f_{\alpha_1}(x_1) \neq 0$. Since C_{α_1} is torsion-free and reduced, it follows from the lemma that there exists $\lambda_1 \in R^*$ such that $f_{\alpha_1}(x_1) \neq 0 \pmod{\lambda_1 C_{\alpha_1}}$. Thus α_1, x_1, λ_1 satisfy conditions (a)–(c). Proceed by induction on n ; assume α_k, x_k , and λ_k have been selected for $k < n$ such that conditions (a)–(c) are satisfied. Then there exist $\beta_1, \dots, \beta_r \in I$ such that, if $\alpha \neq \beta_1, \dots, \beta_r$, then $f_\alpha(x_k) = 0$ for all $k < n$. Since $f_\alpha(A_n) \neq 0$ for infinitely many $\alpha \in I$, we may select $\alpha_n \neq \beta_1, \dots, \beta_r$ such that $f_{\alpha_n}(A_n) \neq 0$. Choose $x_n \in A_n$ such that $f_{\alpha_n}(x_n) \neq 0$; then, since C_{α_n} is torsion-free and reduced, we may again apply the lemma to obtain the existence of $\lambda_n \in R^*$ such that $f_{\alpha_n}(x_n) \neq 0 \pmod{\lambda_n C_{\alpha_n}}$. Since, by choice of α_n , $f_{\alpha_n}(x_k) = 0$ for $k < n$, it is clear that x_k, λ_k, α_k satisfy conditions (a)–(c) for all $k \leq n$.

Write $x_n = (x_n^{(i)})$, where $x_n^{(i)} \in A^{(i)}$. Since $x_n \in A_n$, we have that $x_n^{(i)} = 0$ for $n \geq i$. Hence $x_1^{(i)} = x_1^{(i)} + \lambda_1 x_2^{(i)} + \lambda_1 \lambda_2 x_3^{(i)} + \dots$ is a well-defined element of $A^{(i)}$. Set $x = (x^{(i)}) \in A$; then it is clear that, for any $n \geq 1$, $x = x_1 + \lambda_1 x_2 + \lambda_1 \lambda_2 x_3 + \dots + \lambda_1 \dots \lambda_{n-1} x_n + \lambda_1 \dots \lambda_n y_n$ for some $y_n \in A$.

Now, since C is the direct sum of the C_α , and since $\alpha_n \neq \alpha_k$ for $k < n$, we have that $f_{\alpha_n}(x) = 0$ for some $n \geq 1$. But then $0 = f_{\alpha_n}(x) = f_{\alpha_n}(x_1) + \lambda_1 f_{\alpha_n}(x_2) + \dots + \lambda_1 \dots \lambda_{n-1} f_{\alpha_n}(x_n) + \lambda_1 \dots \lambda_n f_{\alpha_n}(y_n) = \lambda_1 \dots \lambda_{n-1} f_{\alpha_n}(x_n) + \lambda_1 \dots \lambda_n f_{\alpha_n}(y_n)$, since, by condition (b) above, $f_{\alpha_n}(y_k) = 0$ for $k < n$. Therefore, since C_{α_n} is torsion-free, $f_{\alpha_n}(x_n) = -\lambda_n f_{\alpha_n}(y_n)$

$\equiv 0 \pmod{\lambda_n C_{\alpha_n}}$, contrary to condition (c) above. We have thus derived a contradiction, and the proof of the theorem is complete.

COROLLARY 1. *Same hypotheses and notation as in the theorem, with the additional assumption that each $A^{(i)}$ is finitely generated. Then $f_\alpha = 0$ for all but a finite number of $\alpha \in I$.*

PROOF. We may apply the theorem to conclude that, for some $n \geq 1$, $f_\alpha(A_n) \neq 0$ for at most a finite number of $\alpha \in I$. But $A = A' \oplus A_n$, where $A' = \prod_{i=1}^n A^{(i)}$ is finitely generated, since each $A^{(i)}$ is. It then follows immediately that $f_\alpha = 0$ for all but a finite number of $\alpha \in I$, completing the proof.

COROLLARY 2. *Let $\{A^{(i)}\}$ be a sequence of finitely-generated R -modules, and let B be a factor-module of $A = \prod_{i=1}^\infty A^{(i)}$. Suppose that B is a submodule of $C = \sum_{\alpha \in I} C_\alpha$, where each C_α is torsion-free and reduced. Then there exist $\alpha_1, \dots, \alpha_r \in I$ such that $B \subseteq C_{\alpha_1} \oplus \dots \oplus C_{\alpha_r}$.*

PROOF. This follows immediately from Corollary 1.

COROLLARY 3. *Let $\{A^{(i)}\}$ be a sequence of R -modules, and set $\Pi = \prod_{i=1}^\infty A^{(i)}$, $\Sigma = \sum_{i=1}^\infty A^{(i)} \subseteq \Pi$. Let $C = \sum_{\alpha \in I} C_\alpha$, where each C_α is a torsion-free reduced R -module. If $f: \Pi/\Sigma \rightarrow C$ is an R -homomorphism, and f_α is the composition of f with the projection of C onto C_α , then $f_\alpha = 0$ for all but a finite number of $\alpha \in I$. In particular, if $\Pi/\Sigma \subseteq C$, then there exist $\alpha_1, \dots, \alpha_r$ in I such that $\Pi/\Sigma \subseteq C_{\alpha_1} \oplus \dots \oplus C_{\alpha_r}$.*

PROOF. f and f_α may be extended trivially to homomorphisms g and g_α of Π into C and C_α , respectively. Then, by the theorem, there exists $n \geq 1$ such that $g_\alpha(\Pi_n) = 0$ for all but finitely many $\alpha \in I$, where $\Pi_n = \prod_{i=n+1}^\infty A^{(i)}$. Since $\Pi = \Sigma + \Pi_n$ and $g(\Sigma) = 0$, it follows that $g_\alpha = 0$ for all but a finite number of α in I . The corollary follows immediately.

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