ON FUNCTIONS SATISFYING CERTAIN DIFFERENTIAL INEQUALITIES¹

ROBERT J. BUEHLER

1. Introduction. Tagamlitzki [10] has shown that if $f(x)$ is infinitely differentiable for $x > 0$ ($f \in C^\infty(0, \infty)$) and if

$$|f^{(k)}(x)| \leq e^{-x}, \quad x > 0, \quad k = 0, 1, 2, \dots,$$

then $f(x) = ce^{-x}$. In §2 a new proof is given using Liouville's theorem. So far as I am aware, the method of proof does not apply to any essentially more general problems, depending as it does on unique properties of the analytic function e^z . However, I have obtained a few similar results by other methods, and to establish a sufficiently broad framework to contain all cases, it is convenient to pose a somewhat indefinite general problem:

PROBLEM A. Let C_1 be a linear space of functions; let T be an operator on C_1 to C_1 ; and let $\|\cdot\|$ be a norm defined on C_1 . When does $f \in C_1$ and $\|T^k f\| \leq 1$, $k = 0, 1, 2, \dots$, imply that $f \in C_2$ where C_2 is a finite-dimensional subspace of C_1 ?

A narrower framework will also be considered:

PROBLEM B. Let T be a differential operator and let g be a function such that $Tg = -g$ (or $+g$). When does $|T^k f| \leq |g|$, $k = 0, 1, 2, \dots$, imply $f = cg$?

A very simple result of type A is obtained in §3 by taking T to be a self-adjoint operator on a Hilbert space. Differential operators are natural choices for T , and some special cases are spelled out using Fourier and Hermite expansions. A corollary of type B is given.

In §4, I consider the possibility of generalizing results for particular operators T to results for conjugate operators of the form STS^{-1} . For Problem B a natural choice for S is an operator which acts in effect to change the variables of the differential equation $Tf = -f$. By employing this device, results are obtained for example for $T = p_1(x)D + p_0(x)$ and for $T = AD^2 + BD + C$ using respectively Tagamlitzki's theorem and a Fourier expansion.

2. A property of e^{-x} . Theorem 1b is due to Tagamlitzki [10]. For other proofs and generalizations see Tagamlitzki [11; 12], Boas [1], and Obrechhoff [7].

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THEOREM 1a. If $f(z)$ is entire, $\{a_n\}$ is a sequence of positive numbers tending to infinity, and

$$|f^{(k)}(a_n)| \leq e^{-a_n}, \quad n, k = 0, 1, 2, \dots,$$

then $f(z) = ce^{-z}$, where c is constant, $|c| \leq 1$.

PROOF. Taking the absolute value of the Taylor expansion about a_n one obtains

$$|f(z)| \leq \sum |f^{(k)}(a_n)| \cdot |z - a_n|^k / k! \leq e^{-a_n} e^{|z - a_n|}.$$

As $n \rightarrow \infty$, $a_n \rightarrow \infty$, and $-a_n + |z - a_n| \rightarrow -x$ (where $z = x + iy$). Hence for all z , $|f(z)| \leq e^{-x}$. Putting $g(z) = e^z f(z)$, $|g(z)| = |e^z f(z)| = e^x |f(z)| \leq 1$. By Liouville's theorem, $g(z)$ is a constant, and $f(z) = ce^{-z}$.

THEOREM 1b. If $f \in C^\infty(0, \infty)$ and $|f^{(k)}(x)| \leq e^{-x}$ for $x > 0$, $k = 0, 1, 2, \dots$, then $f(x) = ce^{-x}$.

PROOF. The inequalities imply that the Taylor series about an arbitrary point $a > 0$ converges to f ; hence the same proof applies by extending f to an entire function.

3. Some results for self-adjoint operators.

THEOREM 2. Let T be a self-adjoint operator on a Hilbert space C_1 . Let $\{v_n\}$ be a complete orthonormal set such that $Tv_n = \lambda_n v_n$. If for $M > 0$, $f \in C_1$ satisfies

$$(1) \quad \|T^n f\| \leq M, \quad n = n_k, \quad n_k \rightarrow \infty,$$

then $f = \sum_{n \in B} (f, v_n) v_n$ where $n \in B$ if and only if $|\lambda_n| \leq 1$.

The proof can easily be supplied. Some special cases are given in Theorems 3, 4 and 5, using properties of Fourier and Hermite expansions.

THEOREM 3. Let $C_1 = \{f: f \in C^\infty(-\infty, \infty) \text{ and } f \text{ is periodic with period } 2\pi\}$. If $f \in C_1$ and if for some fixed μ, M ,

$$\int_{-\pi}^{\pi} |D^n f|^2 dx \leq M \mu^{2n}, \quad n = n_k, \quad n_k \rightarrow \infty,$$

then $f = \sum_{-\mu}^{[\mu]} a_n e^{in\pi}$.

PROOF. In Theorem 2 take $(f, g) = \int_{-\pi}^{\pi} f \bar{g} \cdot dx$, $v_n = (2\pi)^{-1/2} e^{in\pi}$, $T_n = i\mu^{-1}D$, $\lambda_n = n/\mu$.

COROLLARY 3-1 (CARMICHAEL [4]). Let $f(x)$ be an entire function of exponential type $\leq q$ and let it have period α . Let $m = [\alpha q / 2\pi]$. Then $f(x) = \sum_{-m}^m a_n \exp(2\pi i n x / \alpha)$.

PROOF. Let $f(x) = g(2\pi x/\alpha)$. Then g is of exponential type $\leq b = |\alpha|q/2\pi$, and has period 2π . By Theorem 2.1 of Carmichael [4] for every $\epsilon > 0$, $|g^{(n)}(x)| \leq M(b+\epsilon)^n \exp((b+\epsilon)|x|)$ for $n=0, 1, 2, \dots$, and for some M independent of n and x . Theorem 3 applies with $\mu = b+\epsilon$, and the result follows on letting $\epsilon \rightarrow 0$.

In Theorem 4, the arbitrary interval length l is useful later in Corollary 7-1. Notice that the equations have a much simpler form when $l = \pi$.

THEOREM 4. Let $C_1 = \{f: f \in C^\infty[0, l] \text{ and } f^{(2n)}(0) = f^{(2n)}(l) = 0, n=0, 1, 2, \dots\}$. If for some fixed $M > 0, \mu > 0$,

$$(2) \quad \int_0^l | \{ D^2 + (\pi/l)^2 - 1 \}^n f |^2 dx \leq M \mu^{2n}, \quad n = n_k, \quad n_k \rightarrow \infty,$$

then $f(x) = \sum_1^{[b]} a_n \sin(n\pi x/l)$ where $b^2 = 1 + (l/\pi)^2(\mu - 1)$.

PROOF. Theorem 2 applies with

$$(f, g) = \int_0^l f \bar{g} \cdot dx, \quad T = \mu^{-1} \{ D^2 + (\pi/l)^2 - 1 \},$$

$$v_n = (2/l)^{1/2} \sin(n\pi x/l), \quad \lambda_n = \mu^{-1} \{ (n^2 - 1)(\pi/l)^2 + 1 \}.$$

Taking $l = \pi$ and proceeding as in Corollary 3-1, one obtains the following result of Schoenberg [8].

COROLLARY 4-1. The only functions of exponential type satisfying $f^{(2n)}(0) = f^{(2n)}(\pi) = 0, n=0, 1, 2, \dots$, are the finite sine polynomials $f = \sum_0^k a_n \sin(n\pi x)$ (of type $k\pi$).

Corollary 4-2 is a weaker form of Theorem 4 which will be generalized in §5, and which answers the requirements of Problem B of the introduction, since $f = \sin(\pi x/l)$ satisfies $\{ D^2 + (\pi/l)^2 - 1 \} f = -f$.

COROLLARY 4-2. If $f \in C^\infty[0, l]$ and $| \{ D^2 + (\pi/l)^2 - 1 \}^n f | \leq \sin(\pi x/l)$, then $f(x) = c \cdot \sin(\pi x/l)$.

PROOF. Put $\mu = 1$. The assumed inequalities imply both $f \in C_1$ and (2).

Further special cases of Theorem 2 are obtainable from polynomial expansions. I give here only the Hermite case. Similar results for Laguerre and Jacobi polynomials can be found in [3].

THEOREM 5. Let $C_1 = \{f: f \text{ entire, exponential type}\}$. If $f \in C_1$ and if for some fixed $M > 0, \mu > 0$,

$$\int_{-\infty}^{\infty} e^{-x^2} | (D^2 - 2xD)^n f |^2 dx \leq M \mu^{2n}, \quad n = n_k, \quad n_k \rightarrow \infty,$$

then f is a polynomial of degree $\leq \mu/2$.

PROOF. Theorem 2 applies on taking $T = \mu^{-1}(D^2 - 2xD)$ and $v_n = H_n$ = the normalized Hermite polynomials, which are complete for functions of exponential type (Boas and Buck [2, p. 31]). $Tf \in C_1$ follows from Titchmarsh [13, p. 265], and $Tv_n = -(2n/\mu)v_n$ follows from Szegő [9, p. 102].

4. Conjugacy of operators. Suppose one is able to define on some appropriate space of functions an operator S having C_1 as its range and having a unique inverse S^{-1} defined on C_1 . Then any answer to Problem A for a particular T yields another for the conjugate operator STS^{-1} by Lemma 1.

LEMMA 1. Let $L = STS^{-1}$. Let $SC_1 = \{f: S^{-1}f \in C_1\}$. Define a norm $\|\cdot\|_S$ on SC_1 by $\|f\|_S = \|S^{-1}f\|$. If $f \in C_1$ and $\|T^n f\| \leq 1$ implies $f \in C_2$, then $f_1 \in SC_1$ and $\|L^n f_1\|_S \leq 1$ implies $f_1 \in SC_2$.

PROOF. $1 \geq \|L^n f_1\|_S = \|(STS^{-1})^n f_1\|_S = \|ST^n S^{-1} f_1\|_S = \|S^{-1} ST^n S^{-1} f_1\| = \|T^n S^{-1} f_1\|$. Therefore $S^{-1} f_1 \in C_2$, or $f_1 \in SC_2$.

Possible choices for S include the infinite order differential operators X introduced by Delsarte and Lions [5; 6], which have as range and domain the space of entire functions. Under these operators X all m th order linear differential operators L having leading coefficient equal to unity are conjugate to D^m —that is, $L = XD^m X^{-1}$ for some X . Inserting the Delsarte-Lions operators X for S in Lemma 1, one obtains at least formal generalizations of Theorems 4 and 5; but whether the results can be put into any particularly meaningful or interesting form I do not know.

A type of operator S which applies more naturally to Problem B arises from consideration of familiar transformations of linear differential equations. The equation $Tf \equiv \{\sum p_i(x) D^i\}f = 0$ is transformed into another linear differential equation by substituting (t, v) for (x, f) where

$$(3) \quad v(x) = \phi(x)f(x), \quad t = u(x).$$

It can be shown that (3) is in fact the most general transformation of the form $v = g(x, f)$, $t = h(x, f)$ which preserves linearity; thus it is a natural one to consider. Restricting ϕ and u to C^∞ , and u to be real-valued, the transformation (3) is formalized by what might be called a "multiplication operator" ϕ and a "substitution operator" u defined by

$$(\phi f)(x) = \phi(x)f(x), \quad (uf)(x) = f(u(x)).$$

If ϕ is nonzero and u is monotone, then the inverses $1/\phi$ and u^{-1} exist. It is easily verified that ϕ and u are linear operators and that they satisfy commutation rules such as

$$D\phi = \phi D + \phi', \quad Du = u'D, \quad u\phi = \phi_u u$$

where $\phi_u(x) = \phi(u(x))$.

We now consider the class of operators $S = \phi u$. In discussing the conjugacy relation $L = STS^{-1}$, some attention is needed for domains of definition: if T is defined on $C^\infty(u(a), u(b))$, then L is defined on $C^\infty(a, b)$. Lemma 2 shows how solutions to Problem B can be generalized.

LEMMA 2. Let $L = STS^{-1}$ where $S = \phi u$, and let $h = Sg$ where $Tg = \pm g$. Then: (i) $Lh = \pm h$, and (ii) if $f \in C_1$ and $|T^nf| \leq |g|$ for $u(a) < x < u(b)$ implies $f \in C_2$, then $f \in SC_1$ and $|L^nf_1| \leq |h|$ for $a < x < b$ implies $f_1 \in SC_2$.

PROOF. (i) $Lh = STS^{-1}h = STS^{-1}Sg = STg = \pm Sg = \pm h$. (ii) Define $\|f\| = \sup_{u(a) < x < u(b)} |f/g|$. The result follows from Lemma 1 on writing $\|f\|_S = \|S^{-1}f\| = \sup_{u(a) < x < u(b)} |(u^{-1}\phi^{-1}f)/g| = \sup_{a < x < b} |\phi^{-1}f/ug| = \sup_{a < x < b} |f/\phi ug| = \sup_{a < x < b} |f/h|$.

To what extent are linear differential operators conjugate under the operators $S = \phi u$? Lemma 3 gives an answer for first order operators.

LEMMA 3. Let $L = p_1(x)D + p_0(x)$, where $p_1, p_0 \in C^\infty(a, b)$, and $0 < p_1 < \infty$ for $a < x < b$. Then L is conjugate to D .

PROOF. By commuting the operators, $SDS^{-1} = \phi u D u^{-1} \phi^{-1} = (1/u')(D - \phi'/\phi)$. The requirements $p_1 = 1/u'$ and $p_0 = -\phi'/u'\phi$ are satisfied by putting

$$(4) \quad u(x) = \int_{a'}^x (1/p_1) dx \quad \text{and} \quad \phi(x) = \exp \left(- \int_{a'}^x (p_0/p_1) dx \right)$$

for an arbitrary a' in (a, b) . (Using a' in place of a avoids difficulties when a is a singularity.)

Lemma 4, concerning second order operators, will not actually be used, but it is suggestive of a classification scheme for second order differential equations.

LEMMA 4. Let $T = p_2(x)D^2 + p_1(x)D + p_0(x)$ where $p_0, p_1, p_2 \in C^\infty(a, b)$ and $0 < p_2 < \infty$ for $a < x < b$. There is a function $q(x)$ such that T is conjugate to $D^2 + q$. Moreover q is unique up to translations and reflections of the x axis.

The proof (given in [3]) is similar to that of Lemma 3. The transformation is $D^2 + q = \phi u T u^{-1} \phi^{-1}$ where

$$u^{-1}(x) = \int_c^x p_2^{-1/2} dx, \quad \phi(x) = \exp \int_c^x \{ (p_{1,u} - u'')/2u' \} dx,$$

giving $q = -(\phi''/\phi) + p_{0,u}$, where $p_{i,u}(x) = p_i(u(x))$.

When T has constant coefficients, the canonical form is unique. The operators $4D^2$, $4D^2 + 4$, and $4D^2 + 4D + 4$, for example, are conjugate respectively to D^2 , $D^2 + 4$ (not $D^2 + 1$) and $D^2 + 3$.

5. More general answers to Problem B. Tagamlitzki's theorem may now be generalized by means of Lemmas 2 and 3.

THEOREM 6. Let $f \in C^\infty(a, b)$. Let $L = p_1 D + p_0$ where $p_0, p_1 \in C^\infty(a, b)$ and $0 < p_1 < \infty$ for $a < x < b$. Require further that b be a singular point of L (either $b = \infty$ or $p_1(b) = 0$) such that for any a' in (a, b) ,

$$\int_{a'}^x (1/p_1) dx \rightarrow \infty \quad \text{as } x \rightarrow b.$$

Let h satisfy $Lh = -h$. Then $|L^n f| \leq |h|$, $n = 0, 1, 2, \dots$, $a' < x < b$, implies $f = ch$.

PROOF. By Lemma 3, $L = SDS^{-1}$, where $S = \phi u$ with ϕ and u given by (4). By part (i) of Lemma 2, $Lh = -h$ when $h = Sg$ and $Dg = -g$ (implying, of course, $g = ke^{-x}$). Theorem 1b and part (ii) of Lemma 2 apply with $SC_1 = C^\infty(a', b)$, $C_1 = C^\infty(u(a'), u(b)) = C^\infty(0, \infty)$.

Theorem 7 of Tagamlitzki [11] corresponds to the special case $p_0 = -1$, which gives $h = \text{constant}$.

Next we give an analogous generalization of Corollary 4-2.

THEOREM 7. Let $p_1, p_0 \in C^\infty(a, b)$, $0 < p_1 < \infty$ for $a < x < b$. Define u, l, L , and h by

$$u(x) = \int_a^x (1/p_1) dx, \quad l = u(b), \quad L = (p_1 D + p_0)^2 + (\pi/l)^2 - 1,$$

$$h(x) = \sin\{\pi u(x)/l\} \exp \int_a^x (-p_0/p_1) dx$$

Then: (i) $Lh = -h$, and (ii) $|L^n f| \leq |h|$ for $a < x < b$ implies $f = ch$.

PROOF. Taking $S = \phi u$ from (4) one has by Lemma 3, $(p_1 D + p_0)^2 = (SDS^{-1})^2 = SD^2 S^{-1}$. Hence $L = STS^{-1}$ where $T = D^2 + (\pi/l)^2 - 1$. Since $Tg = -g$ when $g = \sin(\pi x/l)$, (i) follows from Lemma 2; and

(ii) is the result of applying Lemma 2 to Corollary 4-2.

The operator L of Theorem 7 includes as special cases operators having constant coefficients given by

$$(5) \quad L = AD^2 + BD + C \quad \text{where} \quad 1 + C - B^2/4A > 0,$$

the inequality by virtue of $(\pi/l)^2 > 0$. An equivalent restriction is that $Lh = -h$ has an oscillatory general solution, namely,

$$(6) \quad h(x) = c_1 e^{-Bx/2A} \sin\{\pi(x - c_2)/A^{1/2}l\}$$

where

$$l^2 = \pi^2(1 + C - B^2/4A)^{-2}.$$

COROLLARY 7-1. Let L be given by (5), and let h be any solution (6) of $Lh = -h$. If $|L^n f| \leq |h|$ for $n = 0, 1, 2, \dots$, and for all x between any two zeros of h , then $f = ch$.

PROOF. In the theorem put $p_1 = A^{1/2}$, $p_0 = B/2A^{1/2}$, (a, b) = interval between the given zeros of h .

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