ON THE ABSOLUTE HARMONIC SUMMABILITY OF FOURIER SERIES

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1. Introduction. Let $f(x)$ be a continuous periodic function, period 2π , and of bounded variation, and let $\omega(\delta)$ denote the modulus of continuity of f . It is known that [6, pp. 241–242], if $\sum n^{-1}(\omega(n^{-1}))^{1/2}$ is convergent, then the Fourier series of f is absolutely convergent. This result is best possible [2]. In a recent paper [4] Varshney proved that if

$$(1) \quad |f(x+h) - f(x)| \leq A \log^{-1-\epsilon}(1/h) \quad (\epsilon > 0, 0 \leq x \leq 2\pi),$$

then the Fourier series of f is absolutely harmonic summable. In this note we prove the following¹

THEOREM. *If a continuous periodic function $f(x)$, of period 2π , is of bounded variation over $[0, 2\pi]$ and the series*

$$(2) \quad \sum_1^{\infty} (1/n)\omega(1/n)$$

is convergent, then the Fourier series of f is absolutely harmonic summable.

It is easily seen that the condition (1) implies the convergence of the series (2).

2. Lemmas. LEMMA 1.

$$(i) \quad \lim_{x \rightarrow \infty} \omega(1/x) \log x = 0.$$

$$(ii) \quad \sum_1^{\infty} a_n \equiv \sum_1^{\infty} \frac{1}{n2^{n/2}} \left(\int_{1/\pi}^{2^{n+1}/\pi} \omega^2\left(\frac{1}{t}\right) dt \right)^{1/2} < \infty.$$

PROOF. (i) Denote by $A, A_1, \dots, c, \dots$ positive constants. We have for $x > x_0(\epsilon)$,

$$\epsilon > \int_x^{x^2} \frac{1}{t} \omega\left(\frac{1}{t}\right) dt > \omega\left(\frac{1}{x^2}\right) \log x,$$

and (i) follows.

Presented to the Society, January 10, 1961; received by the editors February 23, 1961.

¹ See [1] and [3] for similar results, when f is not of bounded variation.

$$(ii) \quad \int_{1/\pi}^{2^{n+1}/\pi} \omega^2\left(\frac{1}{t}\right) dt < O(1) + \int_{n_0}^{2^{n+1}/\pi} \frac{dt}{(\log t)^2} \\ < O(1) + A \frac{2^n}{n^2}.$$

Hence the convergence of $\sum a_n$ follows by comparison with $\sum 1/n^2$.

LEMMA 2.

$$(i) \quad \sum_1^\infty a_n \equiv \sum_1^\infty \omega(\pi/2^{n+1}) < \infty.$$

$$(ii) \quad \sum_1^\infty b_n \equiv \sum_1^\infty (1/n) \omega^{1/2}(\pi/2^n) < \infty.$$

PROOF. (i) Since the series (2) is convergent, $\sum_1^\infty \omega(1/2^n) < \infty$ and so $\sum a_n < \infty$.

(ii) The series $\sum b_n$ is convergent if

$$\sum_2^\infty (1/n \log n) \omega^{1/2}(1/n) < \infty.$$

Now

$$\sum_2^\infty (1/n \log n) \omega^{1/2}(1/n) = \sum_1 + \sum_2$$

where in $\sum_1$ we put all terms for which $(\omega(1/n))^{1/2} \log n < 1$, and in $\sum_2$ the rest. Then

$$\sum_1 \leq \sum \frac{1}{n(\log n)^2} < \infty, \quad \sum_2 \leq \sum \frac{1}{n} \omega\left(\frac{1}{n}\right) < \infty.$$

LEMMA 3. Let

$$P_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}, \quad n = 0, 1, \dots.$$

Then

$$(i) \quad \sum_2^\infty a_n \equiv \sum_2^\infty \frac{1}{P_{n-1}} \int_0^{1/n} \omega(t) \log(1/t) dt < \infty.$$

$$(ii) \quad \sum_2^\infty b_n \equiv \sum_2^\infty \frac{1}{P_n} \int_0^{1/n} \omega(t) dt < \infty.$$

$$(iii) \quad \sum_2^{\infty} a_n' \equiv \sum_2^{\infty} \frac{1}{n^2 P_{n-1}} \int_{1/n}^{\pi} \frac{\omega(t) dt}{t^2} < \infty.$$

$$(iv) \quad \sum_{n=2}^{\infty} b_n' \equiv \sum_2^{\infty} \frac{1}{(n+1) \log^2 n} \int_{1/n}^{\pi} \frac{\{1 + |\log 1/t|\}}{t} \omega(t) dt < \infty.$$

$$(v) \quad \sum_2^{\infty} \frac{1}{(n+1) \log n} \left(1 - \frac{P_{n-1}}{P_n}\right) \int_{1/n}^{\pi} \omega(t) dt < \infty.$$

PROOF. (i)

$$\begin{aligned} a_n &\leq \frac{A_1}{\log n} \omega\left(\frac{1}{n}\right) \int_n^{\infty} \frac{\log t dt}{t^2} \\ &\leq \frac{A_2}{\log n} \omega\left(\frac{1}{n}\right) \frac{\log n}{n} \end{aligned}$$

and so $\sum a_n < \infty$.

(ii) Since

$$b_n < \frac{A_3}{\log n} \omega\left(\frac{1}{n}\right) \frac{1}{n}, \quad \sum b_n < \infty.$$

(iii)

$$a_n' < \frac{A_4}{n^2 \log n} \int_{1/n}^n \omega\left(\frac{1}{t}\right) dt$$

and

$$\begin{aligned} \sum_2^{\infty} \frac{1}{n^2 \log n} \int_{1/n}^n \omega\left(\frac{1}{t}\right) dt &< \sum_2^{\infty} \frac{1}{n^2 \log n} \left(A_5 + \sum_2^n \omega\left(\frac{1}{k}\right)\right) \\ &< O(1) + \sum_{k=2}^{\infty} \omega\left(\frac{1}{k}\right) \sum_{n=k}^{\infty} \frac{1}{n^2 \log n} \\ &< O(1) + A_6 \sum_{k=2}^{\infty} \omega\left(\frac{1}{k}\right) \frac{1}{k \log k} < \infty. \end{aligned}$$

Hence $\sum a_n' < \infty$.

$$b_n' < \frac{1}{n \log^2 n} \int_{1/n}^n (1 + |\log t|) \frac{\omega(1/t)}{t} dt$$

(iv)

$$< \frac{1}{n \log^2 n} \left(O(1) + \sum_2^n \omega\left(\frac{1}{k}\right) \frac{\log k}{k}\right).$$

Hence

$$\begin{aligned}
\sum b'_n &< O(1) + \sum_2^\infty \frac{1}{n \log^2 n} \sum_2^n \omega\left(\frac{1}{k}\right) \frac{\log k}{k} \\
&< O(1) + \sum_2^\infty \omega\left(\frac{1}{k}\right) \left\{ \frac{\log k}{k} \sum_{n=k}^\infty \frac{1}{n \log^2 n} \right\} \\
&< O(1) + A_7 \sum_2^\infty \frac{1}{k} \omega\left(\frac{1}{k}\right) < \infty.
\end{aligned}$$

The proof of (v) is similar and omitted.

3. Proof of the theorem. Let the Fourier series of $f(t)$ be

$$\frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) \equiv \sum_0^\infty u_n.$$

Set

$$\begin{aligned}
P_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{n+1}, \quad S_n = u_0 + u_1 + \cdots + u_n, \\
t_n &= \frac{(n+1)^{-1}S_0 + n^{-1}S_1 + \cdots + S_n}{P_n}.
\end{aligned}$$

We have to prove the convergence of $\sum |t_n - t_{n-1}|$. Write

$$\begin{aligned}
\phi(t) &= f(x+t) + f(x-t) - 2f(x), \\
\alpha(t) + i\beta(t) &= \sum_{k=0}^\infty \frac{e^{ikt}}{k+1}, \quad 0 < t \leq \pi, \\
\alpha_n &= \int_0^\pi \phi(t) \alpha(t) \cos nt \, dt, \quad \beta_n = \int_0^\pi \phi(t) \beta(t) \sin nt \, dt.
\end{aligned}$$

Then $\phi(t)$ is even and for $t \geq 0$, $|\phi(t)| \leq 2\omega(t)$. Now [1; 4]

$$\begin{aligned}
\pi |t_n - t_{n-1}| &\leq \frac{1}{P_{n-1}} \left| \int_0^\pi \phi(t) \left(\sum_{k=0}^\infty \frac{\cos(n-k)t}{k+1} \right) dt \right| \\
&\quad + \frac{1}{P_{n-1}} \left| \int_0^{1/n} \phi(t) \left(\sum_{k=n}^\infty \frac{\cos(n-k)t}{k+1} \right) dt \right| \\
&\quad + \frac{1}{(n+1)P_n P_{n-1}} \left| \int_0^{1/n} \phi(t) \left(\sum_{k=0}^{n-1} P_k \cos(n-k)t \right) dt \right| \\
&\quad + \frac{1}{P_{n-1}} \left| \int_{1/n}^\pi \phi(t) \left\{ \sum_{k=n}^\infty \frac{\cos(n-k)t}{k+1} + \sum_{k=0}^{n-1} \frac{P_k \cos(n-k)t}{(n+1)P_n} \right\} dt \right| \\
&\equiv T_1(n) + T_2(n) + T_3(n) + T_4(n),
\end{aligned}$$

say. We have, by a lemma due to Tamarkin and Hille [1, p. 182] and Lemma 3,

$$\begin{aligned} \sum_2^\infty T_2(n) &< c_1 \sum_2^\infty \frac{1}{P_{n-1}} \int_0^{1/n} \omega(t) \log \frac{1}{t} dt < \infty, \\ \sum_2^\infty T_3(n) &< \sum_2^\infty \frac{2n}{(n+1)P_n} \int_0^{1/n} \omega(t) dt < \infty, \\ \sum_2^\infty T_4(n) &< c_2 \left\{ \sum_2^\infty \frac{1}{n^2 P_{n-1}} \int_{1/n}^\pi \frac{\omega(t)}{t^2} dt \right. \\ &\quad + \sum_2^\infty \frac{1}{(n+1) \log^2 n} \int_{1/n}^\pi \frac{\omega(t)}{t} \{1 + |\log(1/t)|\} dt \\ &\quad \left. + \sum_2^\infty \frac{1}{\log n} \int_{1/n}^\pi \frac{1}{n+1} \left(1 - \frac{P_{n-1}}{P_n}\right) \omega(t) dt \right\} < \infty, \\ (3) \quad \sum_2^\infty T_1(n) &\leq \sum_2^\infty \frac{|\alpha_n| + |\beta_n|}{P_{n-1}}. \end{aligned}$$

Since $\phi(t)\alpha(t) \in L^2$, we have by Bessel's inequality for the Fourier series of $\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)$ where $h > 0$,

$$\begin{aligned} \sum_1^\infty \alpha_n^2 \sin^2 nh &< c_3 \int_0^\pi \{\phi(t+h)\alpha(t+h) - \phi(t-h)\alpha(t-h)\}^2 dt \\ &< c_4 \left[\int_0^\pi \alpha^2(t+h) \{\phi(t+h) - \phi(t-h)\}^2 dt \right. \\ &\quad + \int_{-h}^h \phi^2(t) \alpha^2(t+2h) dt + \int_{-h}^h \phi^2(t) \alpha^2(t) dt \\ &\quad \left. + \int_h^\pi \phi^2(t) \{\alpha(t+2h) - \alpha(t)\}^2 dt \right] \\ &= c_4 [I_1(h) + I_2(h) + I_3(h) + I_4(h)], \end{aligned}$$

say. Further, when $h \leq \pi/4$,

$$I_2(h) < c_5 h \omega^2(h) \log^2(1/h),$$

$$I_3(h) < c_6 h \omega^2(h) \log^2(1/h).$$

Further, from the expression for $\alpha(t)$ (see also [1, pp. 190–193]),

$$\begin{aligned}
I_4(h) &< c_7 h^2 \int_h^\pi \phi^2(t) \{ \alpha'(t + 2\theta h) \}^2 dt \quad (0 < \theta < 1) \\
&< c_8 h^2 \int_h^\pi \omega^2(t) (t + 2\theta h)^{-2} dt \\
&< c_8 h^2 \int_h^\pi \frac{\omega^2(t)}{t^2} dt = c_8 h^2 \int_{1/\pi}^{1/h} \omega^2\left(\frac{1}{t}\right) dt.
\end{aligned}$$

Now ϕ is continuous and of bounded variation. Hence [4]

$$\begin{aligned}
&\sum_{k=1}^{2N} \left\{ \alpha\left(t + \frac{k\pi}{N}\right) \right\}^2 \left\{ \phi\left(t + \frac{k\pi}{N}\right) - \phi\left(t + \frac{(k-1)\pi}{N}\right) \right\}^2 \\
&< c_9 \{1 + \log^2(1/t)\} \sum_{k=1}^{2N} \left\{ \phi\left(t + \frac{k\pi}{N}\right) - \phi\left(t + \frac{(k-1)\pi}{N}\right) \right\}^2 \\
&< c_9 \{1 + \log^2(1/t)\} \omega_1\left(\frac{\pi}{N}\right) V,
\end{aligned}$$

where V is the total variation of ϕ over $[0, 2\pi]$ and ω_1 denotes the modulus of continuity of ϕ . Since $\omega_1(\delta) \leq 2\omega(\delta)$, we have

$$\begin{aligned}
2N \int_0^\pi \left\{ \alpha\left(t + \frac{\pi}{2N}\right) \right\}^2 \left\{ \phi\left(t + \frac{\pi}{2N}\right) - \phi\left(t - \frac{\pi}{2N}\right) \right\}^2 dt \\
< c_{10} V \left\{ \int_0^\pi \{1 + \log^2(1/t)\} dt \right\} \omega_1\left(\frac{\pi}{N}\right) < c_{11} \omega\left(\frac{\pi}{N}\right).
\end{aligned}$$

Hence, taking $h = \pi/2N$, we have for $N \geq 2$,

$$I_1(h) = I_1(\pi/2N) < c_{11}(1/N)\omega(\pi/N),$$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \alpha_n^2 \sin^2(n\pi/2N) \\
&< c_{12} \left[\frac{1}{N} \omega\left(\frac{\pi}{N}\right) + \frac{1}{N} \omega^2\left(\frac{\pi}{2N}\right) \log^2 N + \frac{1}{N^2} \int_{1/\pi}^{2N/\pi} \omega^2\left(\frac{1}{t}\right) dt \right].
\end{aligned}$$

Hence for $\nu \geq 1$,

$$\begin{aligned}
&\sum_{n=2^{r-1}+1}^{2^r} \alpha_n^2 < 2 \sum_{n=1}^{\infty} \alpha_n^2 \sin^2(n\pi/2^{r+1}) \\
&< c_{13} \left[\frac{1}{2^r} \omega\left(\frac{\pi}{2^r}\right) + \frac{1}{2^r} \omega^2\left(\frac{\pi}{2^{r+1}}\right) \log^2(2^r) + \frac{1}{2^{2r}} \int_{1/\pi}^{2^{r+1}/\pi} \omega^2\left(\frac{1}{t}\right) dt \right].
\end{aligned}$$

Now

$$\left(\sum_1^3 a_i \right)^{1/2} \leq \sum_1^3 a_i^{1/2}, \quad a_i \geq 0; \quad \sum_{n=2^{r-1}+1}^{2^r} (\log n)^{-2} < c_{14} \frac{2^r}{\nu^2}.$$

Hence by an application of Schwarz inequality,

$$\sum_{2^{r-1}+1}^{2^r} \frac{|\alpha_n|}{\log n} < c_{15} \left[\frac{1}{\nu} \omega^{1/2} \left(\frac{\pi}{2^r} \right) + \omega \left(\frac{\pi}{2^{r+1}} \right) + \frac{1}{\nu 2^{r/2}} \left(\int_{1/\pi}^{2^{r+1}/\pi} \omega^2 \left(\frac{1}{t} \right) dt \right)^{1/2} \right],$$

and

$$\begin{aligned} \sum_2^\infty \frac{|\alpha_n|}{\log n} &< c_{15} \left[\sum_1^\infty \frac{1}{\nu} \omega^{1/2} \left(\frac{\pi}{2^r} \right) + \sum_1^\infty \omega \left(\frac{\pi}{2^{r+1}} \right) \right. \\ &\quad \left. + \sum_1^\infty \frac{1}{\nu 2^{r/2}} \left(\int_{1/\pi}^{2^{r+1}/\pi} \omega^2 \left(\frac{1}{t} \right) dt \right)^{1/2} \right] < \infty. \end{aligned}$$

Similarly $\sum_2^\infty |\beta_n|/\log n < \infty$ and so from (3), $\sum_2^\infty T_1(n) < \infty$; and the theorem is proved.

REFERENCES

1. L. McFadden, *Absolute Norlund summability*, Duke Math. J. 9 (1942), 168–207.
2. R. Salem, *On a theorem of Zygmund*, Duke Math. J. 10 (1943), 23–31.
3. O. Szász, *Fourier series and mean moduli of continuity*, Trans. Amer. Math. Soc. 42 (1937), 366–394.
4. O. P. Varshney, *On the absolute harmonic summability of Fourier series*, Proc. Amer. Math. Soc. 11 (1960), 588–595.
5. A. Zygmund, *Sur la convergence absolue des séries de Fourier*, J. London Math. Soc. 3 (1928), 194–196.
6. ———, *Trigonometric series*, vol. 1, Cambridge Univ. Press, New York, 1959.

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