NOTE ON A THEOREM OF NEHARI

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In a recent paper [1] Nehari investigated the oscillation of solutions $y(x)$ of

$$(1) \quad y'' + yF(y^2, x) = 0$$

under the following conditions on the function $F(t, x)$:

$$(2a) \quad F(t, x) \text{ is continuous in } (t, x) \text{ on } \{(t, x): 0 \leq t < \infty, 0 < x < \infty\}.$$

$$(2b) \quad F(t, x) > 0 \text{ for } t > 0, x > 0.$$

$$(2c) \quad \text{For fixed positive } x \text{ and some } \epsilon > 0$$

$$t_2^{-\epsilon} F(t_2, x) > t_1^{-\epsilon} F(t_1, x) \quad (0 \leq t_1 < t_2 < \infty).$$

In the course of this investigation the question arose whether a C^2 solution of (1), with F subject to (2), is uniquely determined by the conditions

$$(C) \quad y(a) = y'(b) = 0, \quad y(x) > 0 \quad \text{for } x \in (a, b].$$

Nehari conjectured that this solution is unique under an additional condition on the behavior of $F(t, x)$ as a function of x . The purpose of this note is to show that such is the case if $F(t, x)$ satisfies the following condition:

$$(2d) \quad \text{For each fixed positive } \rho \text{ and } 0 \leq x_1 < x_2 < \infty, F(\rho, x_2) \geq F(\rho, x_1).$$

We formulate this as a theorem:

THEOREM. *In (1), let $F(t, x)$ satisfy hypotheses (2a) to (2d). Then for each pair (a, b) , $0 \leq a < b < \infty$, there exists a unique solution $y(x)$ of (1) on $[a, b]$ satisfying (C) and having two continuous derivatives on $[a, b]$.*

LEMMA 1. *Let $y_1(x)$ and $y_2(x)$ be two C^2 solutions of (1) on some interval $[a, c]$ such that*

$$(a) \quad y_1(a) = y_2(a) = 0, \quad y_1'(a) = y_2'(a) > 0.$$

$$(b) \quad y_i(x) > 0 \text{ and } y_i'(x) > 0 \text{ for } x \text{ in } (a, c).$$

Then $y_1(x) = y_2(x)$ on $a \leq x < c$.

PROOF. Suppose $y_2(\bar{x}) > y_1(\bar{x})$ for some $\bar{x}$ in (a, c) . Then by the mean-value theorem there exists an x_2 in $(a, \bar{x})$ such that $y_2'(x_2) > y_1'(x_2)$ and by a second application an x_3 in (a, x_2) such that $y_2''(x_3) > y_1''(x_3)$. Because of (2b) and the form of (1), however, this implies $y_2(x_3) < y_1(x_3)$.

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By repeating the foregoing argument one sees that if $y_2(x)$ and $y_1(x)$ differ at any point of (a, c) a situation as in Figure 1 must arise,

FIGURE 1

that is there will exist an x interval $[t_1, t_2] \subset (a, c)$ such that $y_1(t_i) = y_2(t_i)$, $y_2(x) > y_1(x)$ on $t_1 < x < t_2$, and $y_2'(t_1) > y_1'(t_1)$, $y_2'(t_2) < y_1'(t_2)$. We now show that this is impossible.

By the continuity of $(y_2 - y_1)'$ as a function of y , there will exist t_3 and t_4 such that $y_2(t_3) = y_1(t_4) = r$, $y_2'(t_3) = y_1'(t_4) = z$, $t_3 < t_4$ (see Figure 2).

By (1), however, on $t_1 \leq x \leq t_2$

$$(3) \quad y_2(x) = y_2(t_1) + (x - t_1)y_2'(x) + \int_{t_1}^x (s - t_1)y_2(s)F(y_2^2(s), s)ds,$$

$$(4) \quad y_1(x) = y_1(t_1) + (x - t_1)y_1'(x) + \int_{t_1}^x (s - t_1)y_1(s)F(y_1^2(s), s)ds,$$

as is easily seen by differentiating. Using (3) at t_3 and (4) at t_4 and subtracting gives

$$(5) \quad 0 = (t_3 - t_4)z + \int_{t_1}^{t_3} (s - t_1)y_2(s)F(y_2^2(s), s)ds \\ - \int_{t_1}^{t_4} (s - t_1)y_1(s)F(y_1^2(s), s)ds.$$

FIGURE 2

In the first integral make the change of variable $\lambda = y_2(s)$, and in the second $\lambda = y_1(s)$:

$$(6) \quad 0 = (t_3 - t_4)z + \int_p^r (s_2 - t_1)(\lambda) \frac{\lambda}{\lambda_2'(\lambda)} F(\lambda^2, s_2(\lambda)) d\lambda \\ - \int_p^r (s_1 - t_1)(\lambda) \frac{\lambda}{y_1'(\lambda)} F(\lambda^2, s_1(\lambda)) d\lambda.$$

Since for each λ in $[p, r]$ $(s_1 - t_1)(\lambda) \geq (s_2 - t_1)(\lambda)$ and $y_1'(\lambda) \leq y_2'(\lambda)$, while by hypothesis (2d) $F(\lambda^2, s_2(\lambda)) \leq F(\lambda^2, s_1(\lambda))$, the difference between the integrals in (6) is negative. So is the first term, however, so the right side can not be zero. ■

LEMMA 2. Let $y_2(x)$ and $y_1(x)$ be solutions of (1) on $[a, b]$ such that $(i=1, 2)$

$$\begin{aligned} y_i &\in C^2[a, b] \\ y_i(a) &= y_i'(b) = 0 \\ y_i(x) &> 0 \end{aligned} \quad \text{for } x \in (a, b].$$

Then $y_2'(a) = y_1'(a)$.

PROOF. Suppose $y_2'(a) > y_1'(a)$, and consider the function $(y_2 y_1' - y_1 y_2')(x)$. We have

$$(y_2 y_1' - y_1 y_2')' = y_2 y_1 [F(y_2^2, x) - F(y_1^2, x)]$$

so

$$(7) \quad (y_2 y_1' - y_1 y_2')(x) = \int_a^x y_2(s) y_1(s) [F(y_2^2(s), s) - F(y_1^2, s)] ds.$$

By hypothesis $(y_2 y_1' - y_1 y_2')(b) = 0$. The integrand in (7) is positive, however, as long as $y_2(s) > y_1(s)$ and in particular on some interval (a, α) —because $(y_2 - y_1)'(a) > 0$. In fact, α may be taken as b because the same argument as in Lemma 1 shows that the graphs of $y_2(x)$ and $y_1(x)$ can not intersect on (a, b) . Thus the right side of (7) does not tend to zero as $x \rightarrow b^-$. ■

PROOF OF THE THEOREM. The existence of *at least one* solution of (1) + (C) has been proved by Nehari [1, Theorem IV]. By the preceding lemmas there is *at most one* such solution. ■

REFERENCE

1. Zeev Nehari, *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc. **95** (1960), 101–123.

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ON THE MEASURABILITY OF FUNCTIONS IN TWO VARIABLES

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Let (X, μ) and (Y, ν) be two compact spaces having regular Borel measures defined on them. By a measurable modification $\tilde{f}(x, y)$ of a function $f(x, y)$ we mean a function measurable in both variables together and for which $\tilde{f}(x \cdot) = f(x \cdot)$ almost everywhere $[\nu]$ for every x . The purpose of this note is to prove the following theorem.

THEOREM. *If Y is metric and if $f(x, y)$ has a measurable modification and $f(x \cdot)$ is continuous for almost all x , then $f(x, y)$ is measurable in both variables together.*

This theorem was discovered in an effort to prove that the Nelson canonical version [2] is measurable if it has a measurable modification. The theorem would prove this result except for the restriction that Y be metric.

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