ON QUASI-DECOMPOSITIONS OF TORSION FREE ABELIAN GROUPS

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1. Introduction. In this paper we establish a correspondence between quasi-decompositions of certain torsion free Abelian groups and decompositions of their quasi-endomorphism rings into left ideals. (For definitions of these terms, which are taken from [1; 4 and 7], see §2.) The groups we discuss are those whose quasi-endomorphism rings have minimum condition on left ideals and thus include the groups of finite rank. The theorem we obtain is applied in §4 to prove an analog of the Krull-Schmidt Theorem which generalizes a result of Jónsson [4] and to prove an analog of Fitting's Lemma.

The paper had its origin in a suggestion of Professor R. S. Pierce that a proof of Jónsson's Theorem might be based on the discussion of the Krull-Schmidt Theorem given in Jacobson [5] and represents a revision of a part of the author's Ph.D. thesis at the University of Washington. The author would like to thank Professor Pierce for suggesting the problem and for his help and encouragement in the preparation of the paper. He would also like to thank the referee whose suggestions resulted in a substantial simplification of the proofs and a very much improved presentation.

All groups considered are torsion free Abelian groups and we assume that they are subgroups of a fixed rational vector space V . We denote the ring of linear transformations of V by $\mathfrak{L}(V)$. A subgroup G of V will be said to be *full* in V if V/G is a torsion group; equivalently, if V is the minimal divisible group containing G .

2. Preliminaries.

DEFINITION 2.1. Let A and B be any two subgroups of V . We say that

(i) A is *quasi-contained* in B ($A \dot{\subseteq} B$) if $nA \subseteq B$ for some nonzero integer n .

(ii) A and B are *quasi-equal* ($A \dot{=} B$) if $A \dot{\subseteq} B$ and $B \dot{\subseteq} A$.

(iii) A and B are *quasi-isomorphic* ($A \dot{\cong} B$) if there are isomorphic subgroups A' and B' of V such that $A \dot{=} A'$ and $B \dot{=} B'$.

The relations defined on the set of subgroups of V by (ii) and (iii)

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above are equivalence relations and the relation defined in (i) is clearly reflexive and transitive. Furthermore, the subgroups A and B of V are quasi-isomorphic if and only if B is isomorphic to a subgroup of A which is quasi-equal to A . These facts are proved in [1].

DEFINITION 2.2. We will say that the subgroup A of V is *quasi-decomposable* if there exist nonzero independent subgroups B and C of V such that $A \cong B \oplus C$. If A is not quasi-decomposable, A will be said to be *strongly indecomposable*.

DEFINITION 2.3. Let G be a full subgroup of V . Put

$$E(G) = \{\lambda \in \mathfrak{L}(V) \mid \lambda(G) \subseteq G\}.$$

The elements of $E(G)$ will be called *quasi-endomorphisms* of G . If λ is a unit of $\mathfrak{L}(V)$ and both λ and λ^{-1} belong to $E(G)$ then λ will be called a *quasi-automorphism* of G .

It follows from these definitions that $E(G)$ is a subring of $\mathfrak{L}(V)$ and that an element $\lambda \in \mathfrak{L}(V)$ is a quasi-automorphism of G if and only if $\lambda(G) \cong G$. It is also easy to prove.

LEMMA 2.4. Let G and G' be full subgroups of V . If $G \cong G' (G \cong G')$ then $E(G) = E(G') (E(G) \cong E(G'))$.

In Definition 2.3, we have defined $E(G)$ relative to a minimal divisible group V containing G . It is clear that $E(G)$ is uniquely determined up to isomorphism by G . In the lemma below, by a proper idempotent we mean an idempotent $e \neq 0, 1$. The lemma follows directly from the definitions.

LEMMA 2.5. If e is an idempotent element of $E(G)$, then $G \cong H_1 \oplus H_2$ where $H_1 = e(V) \cap G \cong e(G)$ and $H_2 = (1-e)(V) \cap G \cong (1-e)(G)$. Conversely, if $G \cong H_1 \oplus H_2$, there is a unique idempotent $e \in E(G)$ such that $H_1 \cong e(G)$, $H_2 \cong (1-e)(G)$. Thus, G is strongly indecomposable if and only if $E(G)$ contains no proper idempotents.

3. The Main Theorem. Where proofs have been omitted below, the results are either well known [6, pp. 50–51] or follow easily from well-known facts.

LEMMA 3.1. Let e and f be nonzero idempotents in $E(G)$. Let $W = e(V)$, $U = f(V)$. Let $E(e(G), f(G)) = \{\phi \in \text{Hom}(W, U) \mid \phi(e(G)) \subseteq f(G)\}$. Then the restriction mapping $\lambda \rightarrow \lambda|_W$ sends $fE(G)e$ one-to-one onto $E(e(G), f(G))$.

PROOF. Put $W' = (1-e)(V)$. It is easy to see that the elements of $fE(G)e$ are just those $\lambda \in \mathfrak{L}(V)$ such that $\lambda|_W \in E(e(G), f(G))$ and $\lambda|_{W'} = 0$. This proves the lemma.

THEOREM 3.2. *Let e be a nonzero idempotent of $E(G)$. Then the rings $E(e(G))$ and $eE(G)e$ are isomorphic to each other and anti-isomorphic to $\text{Hom}_{E(G)}(E(G)e, E(G)e)$.*

PROOF. That $E(e(G)) \cong eE(G)e$ follows from Lemma 3.1 and the observation that $E(e(G)) \cong E(e(G), e(G))$. The second statement is well-known [6, p. 51].

COROLLARY 3.3. *Let e be a nonzero idempotent of $E(G)$. Then the following are equivalent conditions: (i) $e(G)$ is strongly indecomposable; (ii) $eE(G)e$ has no proper idempotents; (iii) $E(G)e$ is indecomposable as an $E(G)$ module.*

PROOF. The equivalence of (i) and (ii) follows from Theorem 3.2 and Lemma 2.5. That (ii) and (iii) are equivalent is well known.

THEOREM 3.4. *Let e and f be nonzero idempotents of $E(G)$. Then $e(G) \cong f(G)$ if and only if $E(G)e$ is isomorphic to $E(G)f$ as an $E(G)$ module.*

PROOF. It is well known [6, p. 51] that $E(G)e \cong E(G)f$ if and only if u, v exist in $E(G)$ such that $u \in eE(G)f$, $v \in fE(G)e$, $e = euv$ and $f = fvu$. By Lemma 3.1 this is equivalent to the existence of $\phi \in E(f(G), e(G))$ and $\psi \in E(e(G), f(G))$ such that $\phi = \psi^{-1}$, or in other words, to $e(G) \cong f(G)$.

THEOREM 3.5. *Suppose that $E(G)$ has the minimum condition on left ideals. Then any decomposition $G \cong \sum_{\lambda \in \Lambda} \oplus H_\lambda$ of G into nonzero subgroups has only finitely many summands. The correspondence*

$$G \cong \sum_{i=1}^k \oplus e_i G \rightarrow E(G) = \sum_{i=1}^k \oplus E(G)e_i \quad (e_i \text{ idempotent})$$

between the decompositions of G and the decompositions of $E(G)$ into a direct sum of left ideals is one-to-one.

PROOF. Let $G \cong \sum_{\lambda \in \Lambda} \oplus H_\lambda$. By Lemma 2.4, $E(G) = E(\sum_{\lambda \in \Lambda} \oplus H_\lambda)$ so the projections e_λ of this decomposition belong to $E(G)$. Since $1 \in E(G)$ and for each λ , H_λ is nonzero, it follows that $L_\lambda = E(G)e_\lambda$ is a nonzero left ideal of $E(G)$. It is clear that the sum $\sum_{\lambda \in \Lambda} L_\lambda$ is direct. Minimum condition on $E(G)$ implies that Λ is finite. We now have $1 = \sum_{\lambda \in \Lambda} e_\lambda$ since this is true on $\sum_{\lambda \in \Lambda} \oplus H_\lambda$ which is full in V . Thus, $E(G) = \sum_{\lambda \in \Lambda} \oplus L_\lambda$.

Now suppose that $E(G) = \sum_{i=1}^k \oplus L_i$ with each L_i a nonzero left ideal of $E(G)$. Then, as is well known, there exist orthogonal idempotents e_i such that $L_i = E(G)e_i$ ($i = 1, \dots, k$). By an obvious ex-

tension of Lemma 2.5 we have $G \doteq \sum_{i=1}^t \oplus e_i(G)$. It is now clear that the correspondence is one-to-one.

4. Applications. In this section we assume that $E(G)$ has the minimum condition on left ideals.

THEOREM 4.1. *Let $\sum_{i=1}^s \oplus H_i \doteq G \doteq \sum_{j=1}^t \oplus K_j$ with H_i and K_j strongly indecomposable ($i=1, \dots, s$; $j=1, \dots, t$). Then $s=t$ and for some permutation π of $\{1, \dots, t\}$, we have $H_i \cong K_{\pi(i)}$ ($i=1, \dots, t$).*

PROOF. Let $E(G) = \sum_{i=1}^s \oplus L_i$ and $E(G) = \sum_{j=1}^t \oplus M_j$ be the decompositions of $E(G)$ associated with the given decompositions of G by Theorem 3.5. Then, by Corollary 3.3, the L_i and M_j are indecomposable $E(G)$ modules. Since $E(G)$ has minimum condition on left ideals and the additive group of $E(G)$ is torsion free, E also has maximum condition on left ideals (see [2, p. 285]). By the Krull-Schmidt Theorem for operator groups, we conclude that $s=t$ and for each i there is an $E(G)$ -isomorphism mapping L_i onto $M_{\pi(i)}$ where π is some permutation of $\{1, \dots, t\}$. By Theorem 3.4 this implies that $H_i \cong K_{\pi(i)}$ ($i=1, \dots, t$) and the proof is complete.

If the group G has finite rank then $E(G)$ is a finite dimensional rational algebra and so has minimum condition on left ideals. This is the case treated in [4]. However, it follows easily from a theorem of L. Fuchs [3] that there exist torsion free groups G of infinite rank such that $E(G)$ is a finite dimensional rational algebra and Theorem 4.1 applies to these groups as well.

THEOREM 4.2. *For any $\lambda \in E(G)$ there exists a decomposition $G \doteq H_1 \oplus H_2$ such that λ induces a quasi-automorphism on H_1 and λ induces a nilpotent quasi-endomorphism on H_2 .*

PROOF. The mapping $\lambda^*: x \rightarrow x\lambda$ is an $E(G)$ -endomorphism of the left $E(G)$ -module $E(G)$ so that Fitting's Lemma for operator groups yields a module decomposition $E(G) = L_1 \oplus L_2$ such that λ^* is an automorphism on L_1 and is nilpotent on L_2 . Let $L_i = E(G)e_i$ for orthogonal idempotents e_i ($i=1, 2$). By Lemma 2.5 we have $G \doteq H_1 \oplus H_2$ with $H_i \doteq e_i(G)$.

Now, $\lambda e_i \in Ee_i = L_i$ and $e_i\lambda \in L_i\lambda \subseteq L_i$ ($i=1, 2$). Comparing the equations $\lambda = \lambda \cdot 1 = \lambda e_1 + \lambda e_2$ and $\lambda = 1 \cdot \lambda = e_1\lambda + e_2\lambda$ we have $e_i\lambda = \lambda e_i$ ($i=1, 2$). Thus, $e_i\lambda = e_i\lambda e_i \in e_i E(G) e_i$ ($i=1, 2$). The theorem follows easily now from Lemma 3.2, Theorem 3.3 and the well-known fact that the elements of $\text{Hom}_{E(G)}(E(G)e, E(G)e)$ are just the right multiplications by elements of $eE(G)e$.

COROLLARY 4.3. *If G is strongly indecomposable then any quasi-endomorphism of G is either nilpotent or is a quasi-automorphism.*

COROLLARY 4.4. *If G is strongly indecomposable then the sum of two nilpotent quasi-endomorphisms is again nilpotent.*

PROOF. As in Jacobson [5, p. 156].

One can use the corollaries above to show that the radical of $E(G)$ with G strongly indecomposable coincides with the set of nilpotent elements. We remark finally that the results above can be extended to torsion free modules over an integral domain. We note, however, that the theorem of Fuchs referred to after the proof of Theorem 4.1 uses the Unique Factorization Theorem for the integers and, to the author's knowledge, would not be available should such an extension be made.

BIBLIOGRAPHY

1. R. A. Beaumont and R. S. Pierce, *Torsion free rings*, Illinois J. Math. **5** (1961), 61–98.
2. L. Fuchs, *Abelian groups*, Hungarian Academy of Sciences, Budapest, 1958.
3. ———, *Indecomposable Abelian groups of arbitrary power*, Acta Math. Acad. Sci. Hungar. **10** (1959), 453–457.
4. B. Jónsson, *On direct decompositions of torsion free Abelian groups*, Math. Scand. **7** (1959), 361–371.
5. N. Jacobson, *Lectures in abstract algebra, Basic concepts*, Vol. 1, Van Nostrand, New York, 1951.
6. ———, *Structure of rings*, Amer. Math. Soc. Colloq. Publ. Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.
7. R. S. Pierce, *Subrings of simple algebras*, Michigan Math. J. **7** (1960), 241–243.

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