ARITHMETICAL NOTES. VIII. AN ASYMPTOMATIC FORMULA OF RÉNYI¹

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1. **Introduction.** A prime divisor p of a positive integer n will be said to be of multiplicity k if $p^k | n$, $p^{k+1} \nmid n$. In the cases $k=1$, $k=2$, $k \geq 2$, prime divisors of multiplicity k will be referred to, respectively, as *simple*, *double*, and *multiple* prime divisors. The set Q of the square-free integers consists of the integers n whose prime divisors are all simple. A related set is the sequence K of those integers n whose prime divisors are simple, with the exception of a single double prime divisor.

The set of integers K occurs in the evaluation of the inversion function of the sequence of primes (cf. [1, §2] and the references indicated there). This set also plays an important part in determining the number of solutions of certain congruences involving prime summands (cf. [2, §2]).

Let now $K(x)$ denote the enumerative function of K , that is, the number of integers $n \leq x$ contained in K . The asymptotic density δ of K has been determined by Rényi, who showed [4, (6)] that

$$(1.1) \quad \delta \equiv \lim_{x \rightarrow \infty} \frac{K(x)}{x} = \frac{6\alpha}{\pi^2}, \quad \alpha \equiv \sum_p \frac{1}{p(p+1)},$$

where the summation ranges over all primes p .

Rényi's method does not, however, yield an estimate for $K(x)$ with a remainder term. It is the purpose of this note to refine Rényi's result by proving the following

THEOREM. *If $x \geq 3$, then*

$$(1.2) \quad K(x) = \frac{6\alpha x}{\pi^2} + O(x^{1/2} \log \log x),$$

where α is defined by (1.1).

A wide extension of this result is proved in a paper to appear elsewhere. The separate consideration of the special case of the present paper seems justified by virtue of its particular interest and also by

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the somewhat simpler details of proof.

2. **Preliminaries.** Let $\mu(n)$ denote the Möbius inversion function. We recall the familiar identities (cf. [3 Theorems 287 and 302]),

$$(2.1) \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2},$$

$$(2.2) \quad \sum_{d^2 \delta = n} \mu(d) = q(n),$$

where $q(n) = 1$ or 0 according as n is or is not in Q .

We shall also need the identity

LEMMA 2.1. *If p is a fixed prime, then*

$$(2.3) \quad S_p \equiv \sum_{\substack{n=1 \\ p \nmid n}}^{\infty} \frac{\mu(n)}{n^2} = \left(\frac{p^2}{p^2 - 1} \right) \frac{6}{\pi^2}.$$

PROOF. By the multiplicative property of $\mu(n)$ and (2.1),

$$S_p = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} - \sum_{m=1}^{\infty} \frac{\mu(pm)}{(pm)^2} = \frac{6}{\pi^2} + \frac{1}{p^2} \sum_{\substack{m=1 \\ (p,m)=1}}^{\infty} \frac{\mu(m)}{m^2},$$

so that $S_p = 6/\pi^2 + p^{-2}S_p$. This completes the proof.

Except for the trivial estimate, $[x] = x + O(1)$, where $[x]$ denotes the greatest integer $\leq x$, we shall need only the familiar approximations,

$$(2.4) \quad \sum_{n > x} \frac{1}{n^2} = O\left(\frac{1}{x}\right),$$

$$\sum_{p \leq x} \frac{1}{p} = O(\log \log x),$$

where the second summation is over the primes $p \leq x$ ($x \geq 3$).

3. **The proof.** In his proof of (1.1), Rényi made use of the simple observation that every n has a unique factorization of the form, $n = n_1 n_2$, where $(n_1, n_2) = 1$, $n_2 \in Q$, and n_1 has only multiple prime divisors. Rényi actually considered a much more general problem. In the special problem under discussion, the above uniqueness property can be replaced by the following simpler observation:

REMARK 3.1. Every n has a unique representation of the form, $n = c^2 e$ where $c > 0$ and $e \in Q$.

PROOF OF THEOREM. For an arbitrary prime $p \leq x^{1/2}$, let $K_p(x)$ de-

note the number of $n \leq x$ contained in K with p^2 as a factor. It follows from the above remark and (2.2) that

$$\begin{aligned} K_p(x) &= \sum_{\substack{p^3 e \leq x \\ (p, e)=1}} q(e) = \sum_{\substack{e \leq x/p^3 \\ (p, e)=1}} \sum_{d^2 e = e} \mu(d) = \sum_{\substack{d^2 \delta \leq x/p^3 \\ (p, d\delta)=1}} \mu(d) \\ &= \sum_{\substack{d \leq x^{1/2}/p \\ p \nmid d}} \mu(d) \sum_{\substack{\delta \leq x/p^3 d^2 \\ p \nmid \delta}} 1 = \sum_{\substack{d \leq x^{1/2}/p \\ p \nmid d}} \mu(d) \left(\left[\frac{x}{p^2 d^2} \right] - \left[\frac{x}{p^3 d^2} \right] \right); \end{aligned}$$

hence

$$(3.1) \quad K_p(x) = \sum_{\substack{d \leq x^{1/2}/p \\ p \nmid d}} \mu(d) \left(\frac{x}{d^2} \left(\frac{1}{p^2} - \frac{1}{p^3} \right) + O(1) \right) \equiv \sum_1 + \sum_2,$$

where

$$(3.2) \quad \sum_1 = x \left(\frac{p-1}{p^3} \right) \sum_{\substack{d \leq x^{1/2}/p \\ p \nmid d}} \frac{\mu(d)}{d^2}, \quad \sum_2 = O \left(\sum_{d \leq x^{1/2}/p} 1 \right) = O \left(\frac{x^{1/2}}{p} \right),$$

the O -constant being uniform for all $p \leq x^{1/2}$. Therefore

$$\sum_1 = x \left(\frac{p-1}{p^3} \right) \sum_{\substack{d=1 \\ p \nmid d}}^{\infty} \frac{\mu(d)}{d^2} + O \left(\frac{x}{p^2} \sum_{d > x^{1/2}/p} \frac{1}{d^2} \right),$$

so that by (2.3) and (2.4), we have the uniform estimate,

$$(3.3) \quad \sum_1 = \frac{6x}{p(p+1)\pi^2} + O \left(\frac{x^{1/2}}{p} \right).$$

Hence it follows from (3.1), (3.2), and (3.3) that

$$(3.4) \quad K_p(x) = \frac{6x}{p(p+1)\pi^2} + O \left(\frac{x^{1/2}}{p} \right),$$

uniformly in $p \leq x^{1/2}$.

Now

$$(3.5) \quad K(x) = \sum_{p \leq x^{1/2}} K_p(x),$$

where the summation is over all primes $\leq x^{1/2}$. Hence by (3.4), (3.5), and (2.4),

$$\begin{aligned}
 K(x) &= \frac{6x}{\pi^2} \sum_{p \leq x^{1/2}} \frac{1}{p(p+1)} + O\left(\sum_{p \leq x^{1/2}} \frac{x^{1/2}}{p}\right) \\
 &= \frac{6x}{\pi^2} \sum_{p=1}^{\infty} \frac{1}{p(p+1)} + O\left(x \sum_{n > x^{1/2}} \frac{1}{n^2}\right) + O\left(x^{1/2} \sum_{p \leq x} \frac{1}{p}\right) \\
 &= \frac{6\alpha x}{\pi^2} + O(x^{1/2}) + O(x^{1/2} \log \log x) = \frac{6\alpha x}{\pi^2} + O(x^{1/2} \log \log x).
 \end{aligned}$$

This completes the proof.

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