A LIMIT THEOREM FOR THE FRACTIONAL PARTS OF THE SEQUENCE $\{2^i t\}$

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1. **Introduction.** Let

$$(1.1) \quad x_n(t) = \frac{1}{n^{1/2}} \sum_{i=0}^{n-1} [f_i(2^i x) - t]$$

for $n = 1, 2, \dots$, where

$$(1.2) \quad f_t(x) = \begin{cases} 1 & \text{for } 0 < x < t, \\ 0 & \text{for } t \leq x \leq 1, \end{cases}$$

and $f_t(x+1) = f_t(x)$ ($0 \leq t \leq 1$). For fixed n we consider $\{x_n(t)\}$ as a stochastic process with the time interval $\langle 0, 1 \rangle$ and multivariate distribution defined as follows:

$$(1.3) \quad \begin{aligned} &P\{x_n(t_1) < \omega_1, \dots, x_n(t_s) < \omega_s\} \\ &= m\{x: 0 \leq x \leq 1; x_n(t_i) < \omega_i, i = 1, \dots, s\}, \end{aligned}$$

where m denotes the Lebesgue measure.

It was proved by N. J. Fine [2] that for given $t_i \in \langle 0, 1 \rangle$ and ω_i (s any positive integer; for $s=1$ see [3]), the probability (1.3) approaches

$$(1.4) \quad P\{x(t_1) < \omega_1, \dots, x(t_s) < \omega_s\}$$

as $n \rightarrow \infty$. Here $\{x(t)\}$, $0 \leq t \leq 1$, is a Gaussian process associated with the covariance

$$(1.5) \quad \rho(t, s) = \lim_{n \rightarrow \infty} \int_0^1 x_n(t) x_n(s) dx$$

and with $E\{x(t)\} = 0$.

The purpose of this paper is to prove the following limit theorem, which was conjectured by M. Kac:

$$(1.6) \quad \lim_{n \rightarrow \infty} \left\{ \sup_{0 \leq t \leq 1} |x_n(t)| < \omega \right\} = P \left\{ \max_{0 \leq t \leq 1} |x(t)| < \omega \right\}.$$

Actually we prove convergence in distribution for a wide class of functionals of the sample functions by using a general invariance theorem of Skorokhod [5]. $\sup_{0 \leq t \leq 1} (\cdot)$ is only a specific example of the result.

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2. Preliminary results. In our proof we shall use the following results.

Let¹ K be the space of all left continuous functions on $\langle 0, 1 \rangle$ which also have right limits at every point and are right continuous at zero.

The topology J in K is determined by the following convergence: the sequence $y_n(t)$ converges to $y_0(t)$ if there exists a sequence of monotonic continuous functions $\lambda_n(t)$ for which

$$(2.1) \quad \begin{aligned} \lambda_n(0) = 0, \quad \lambda_n(1) = 1, \quad \lim_{n \rightarrow \infty} \max_{0 \leq t \leq 1} |\lambda_n(t) - t| &= 0, \\ \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq 1} |y_n(\lambda_n(t)) - y_0(t)| &= 0. \end{aligned}$$

LEMMA 1, [5]. Let $\{\xi_n(t)\}$, $0 \leq t \leq 1$, $n=0, 1, \dots$, be a sequence of stochastic processes such that $\xi_n(t)$, $n=0, 1, \dots$, are in K with probability one. Then, the distribution of $F(\xi_n)$ approaches the distribution of $F(\xi_0)$ for every real functional $F(\cdot)$, defined on K , and continuous in the topology J , if and only if

(a) The multivariate distribution of $[\xi_n(t_1), \dots, \xi_n(t_s)]$ approaches the multivariate distribution of $[\xi_0(t_1), \dots, \xi_0(t_s)]$ for all s , and $t_1, \dots, t_s$ from some set N which is dense in $\langle 0, 1 \rangle$,

(b) for each $\epsilon > 0$

$$\lim_{\epsilon \downarrow 0} \lim_{n \rightarrow \infty} P \left\{ \sup_{t-\epsilon < t_1 < t < t_2 < t+\epsilon} \min [|\xi_n(t_1) - \xi_n(t)|, |\xi_n(t) - \xi_n(t_2)|] > \epsilon \right\} = 0.$$

LEMMA 2 (S. BERNSTEIN, see [4, p. 19]). Let $\{Y_k\}$, $k=1, \dots, n$, be independent binomial random variables such that $P\{Y_k=1\}=p$, $P\{Y_k=0\}=q$. Then

$$P \left\{ \left| \sum_{k=1}^n (Y_k - p) \right| \geq 2\omega(npq)^{1/2} \right\} \leq 2e^{-\omega^2}$$

for $0 \leq \omega \leq (3/2)(npq)^{1/2}$.

LEMMA 3. Let $x_n(t)$ be defined by (1.1) and (1.2) and let $n > m \geq 0$, $1 \leq i \leq 2^m$. Then for $0 \leq \omega \leq 3(n-m)/2^{m+1/2}n^{1/2}$

$$P \left\{ \left| x_n \left(\frac{i}{2^m} \right) - x_n \left(\frac{i-1}{2^m} \right) \right| > \omega \right\} \leq 2m \exp \left(-\frac{\omega^2 2^m}{8(m+1)} \right).$$

PROOF. We can assume $m \geq 1$ because for $m=0$ the inequality is

¹ We take left continuity instead of right continuity as required by Skorokhod [5]. This clearly does not change the validity of Skorokhod's result but is somewhat more convenient for the proof of Lemma 3.

trivial. Let $I = \langle i/2^m, (i-1)/2^m \rangle$, let x have a uniform distribution on $\langle 0, 1 \rangle$, and let

$$(2.2) \quad Y_k = Y_k(x) = \begin{cases} 1 & \text{if } 2^k x \pmod{1} \in I, \\ 0 & \text{if } 2^k x \pmod{1} \notin I. \end{cases}$$

If x has the diadic expansion

$$x = \sum_{r=1}^{\infty} \epsilon_r(x)/2^r$$

then

$$2^k x \pmod{1} = \sum_{r=k+1}^{\infty} 2^{k-r} \epsilon_r(x)$$

and Y_k depends only on the digits $\epsilon_{k+1}, \dots, \epsilon_{k+m}$. Thus, for every k , $Y_k, Y_{k+m}, Y_{k+2m}, \dots$ are independent binomial variables with probability of success $p = 2^{-m}$. By Lemma 2 we get

$$(2.3) \quad P \left\{ \left| \sum_{s=0}^r (Y_{ms+k} - p) \right| > 2\omega((r+1)p)^{1/2} \right\} \leq 2e^{-\omega^2}$$

for $0 \leq \omega \leq (3/2^{3/2})((r+1)p)^{1/2}$. If we apply (2.3) with $r = r_k = [n/m]$ for $k \leq n - m[n/m]$ and $r = r_k = [n/m] - 1$ for $k > n - m[n/m]$ (here $[x]$ denotes the greatest integer $\leq x$) we get

$$\begin{aligned} P \left\{ \left| \sum_{k=1}^n (Y_k - p) \right| > \omega \right\} &\leq \sum_{k=1}^m P \left\{ \left| \sum_{s=0}^{r_k} (Y_{ms+k} - p) \right| > \frac{\omega}{m} \right\} \\ &\leq 2m \exp \left(- \frac{\omega^2}{4m(n+m)p} \right) \end{aligned}$$

for $0 \leq \omega \leq (3/2^{1/2})(p(n-m))$. Finally, according to (2.2) we have

$$x_n \left(\frac{i}{2^m} \right) - x_n \left(\frac{i-1}{2^m} \right) = \frac{1}{n^{1/2}} \sum_{k=1}^n (Y_k - p),$$

hence our lemma follows.

3. The proof of (1.6). Let us put in Lemma 1 $\xi_n(t) = x_n(t)$, $n = 1, 2, \dots$, and $\xi_0(t) = x(t)$. It is clear that $x_n(t)$ according to (1.1) and (1.2) is left continuous (in t), has right hand limits and is right continuous at zero. It was also proved in [1] that $x(t)$ is continuous with probability one. Condition (a) of Lemma 1 is satisfied by Fine's result [2]. Thus, if we show that condition (b) is also satisfied then

the convergence in distribution for any functional F on K , continuous in the topology J , follows from Lemma 1. Clearly the functional

$$F_0(y) = \sup_{0 \leq t \leq 1} |y(t)|, \quad y(\cdot) \in K$$

is continuous in the topology J (cf. (2.1)). We may write $\max$ instead of $\sup$ in the right hand side of (1.6) because $x(t)$ is continuous with probability one.

Let now M be fixed integer and put

$$N = \left[\frac{2}{3} \log_2 n \right].$$

Let $1 \leq k \leq 2^N$, $1 \leq u \leq 2^M$ and

$$(3.1) \quad \frac{u-1}{2^M} \leq \frac{k}{2^N} < \frac{u}{2^M}.$$

We can then find $\eta(M+1), \eta(M+2), \dots, \eta(N), \eta=0, 1$ such that

$$\frac{k}{2^N} = \frac{u-1}{2^M} + \sum_{m=M+1}^N \frac{\eta(m)}{2^m}.$$

Consequently, if we define u_m by

$$u_m = \frac{u-1}{2^M} + \sum_{j=M+1}^m \frac{\eta(j)}{2^j}, \quad u_M = \frac{u-1}{2^M}$$

then

$$(3.2) \quad \begin{aligned} \left| x_n\left(\frac{k}{2^N}\right) - x_n\left(\frac{u-1}{2^M}\right) \right| &\leq \sum_{m=M+1}^N \left| x_n(u_{m-1}) - x_n\left(u_{m-1} + \frac{1}{2^m}\right) \right| \\ &\leq \sum_{m=M+1}^N \max_{1 \leq i \leq 2^m} \left| x_n\left(\frac{i}{2^m}\right) - x_n\left(\frac{i-1}{2^m}\right) \right|. \end{aligned}$$

Moreover, for $(k-1)/2^N \leq t < k/2^N$, by (1.1) and (1.2)

$$(3.3) \quad \begin{aligned} \left| x_n(t) - x_n\left(\frac{k-1}{2^N}\right) \right| &\leq \left| x_n\left(\frac{k}{2^N}\right) - x_n\left(\frac{k-1}{2^N}\right) \right| + \frac{n^{1/2}}{2^N} \\ &\leq \left| x_n\left(\frac{k}{2^N}\right) - x_n\left(\frac{k-1}{2^N}\right) \right| + 2n^{-1/6}. \end{aligned}$$

Combining (3.1)–(3.3), we have

$$(3.4) \quad \sup_{|t_1 - t_2| \leq 2^{-M-1}} |x_n(t_1) - x_n(t_2)| \leq 4n^{-1/6} + 4 \sum_{m=M}^N \max_{1 \leq i \leq 2^m} \left| x_n\left(\frac{i}{2^m}\right) - x_n\left(\frac{i-1}{2^m}\right) \right|.$$

However, by Lemma 3, we have for any $0 \leq \omega \leq 3/2 \cdot 2^{1/2}$ and $m \leq 2 \log_2 n / 3$

$$P \left\{ \max_{1 \leq i \leq 2^m} \left| x_n\left(\frac{i}{2^m}\right) - x_n\left(\frac{i-1}{2^m}\right) \right| > \omega 2^{-m/4} \right\} \leq 2^{m+1} m \exp \left(- \frac{\omega^2 2^{m/2}}{8(m+1)} \right)$$

and therefore

$$(3.5) \quad \sum_{m=M}^N \max_{1 \leq i \leq 2^m} \left| x_n\left(\frac{i}{2^m}\right) - x_n\left(\frac{i-1}{2^m}\right) \right| \leq \omega \sum_{m=M}^N 2^{-m/4}$$

except for a set of probability not exceeding

$$(3.6) \quad \sum_{m=M}^{\infty} 2^{m+1} m \exp \left(- \frac{\omega^2 2^{m/2}}{8(m+1)} \right).$$

Since both (3.6) and the right hand side of (3.5) tend to zero as $M \rightarrow \infty$, condition (b) follows from (3.4). This completes the proof.

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