DIRECT DECOMPOSITIONS OF LATTICES OF CONTINUOUS FUNCTIONS

ROBERT L. BLAIR AND CLAUDE W. BURRILL

If X is a topological space and if K is a chain equipped with its order topology, then we denote by $C(X, K)$ the lattice of all continuous functions from X to K. If X is the union of two disjoint open-and-closed subsets X_1 and X_2, then it is clear that $C(X, K)$ is isomorphic to the direct product of the lattices $C(X_1, K)$ and $C(X_2, K)$. In Theorem 2 of [2], Kaplansky proves the following converse:

THEOREM A (KAPLANSKY). *If X is compact, if K has neither a first nor a last element, and if $C(X, K)$ is isomorphic to the direct product of two lattices L_1 and L_2, then X is the union of disjoint open-and-closed subsets X_1 and X_2 having the property that L_i is isomorphic to $C(X_i, K)$ $(i=1, 2)$.*

A technique for removing the stated hypothesis on K is outlined in §6 of [2]. The validity of Theorem A for noncompact spaces, however, is left as an open question in [2].[1] In this note we shall remove from Theorem A both the hypothesis on K and the compactness hypothesis on X.[2] At the same time, we shall show that a direct decomposition of merely a *sublattice* of $C(X, K)$ (satisfying a very mild condition) is enough to ensure a corresponding decomposition of X (Theorem B below). The sublattices that we find adequate for this purpose are described as follows (cf. the concluding remark of this note):

DEFINITION. A sublattice L of $C(X, K)$ will be called *adequate* in case for each $x \in X$ there exist functions $f, g \in L$ such that $f(x) < g(x)$.

For example, if L is a sublattice of $C(X, K)$ that contains at least two distinct constant functions, then obviously L is adequate.

By a *prime ideal* of a lattice L we mean a nonempty proper subset P of L such that (i) if $a, b \in P$, then $a \vee b \in P$ and (ii) $a \wedge b \in P$ if and only if $a \in P$ or $b \in P$; a *dual prime ideal* is the complement of a prime ideal (see e.g. [1]). We require the following readily verified fact (cf.

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[1] If K is the chain R of real numbers, then (as observed in [2, p. 621]) a reduction to the compact case is possible via the Stone-Čech compactification (of a suitable completely regular space). One should note, however, that this device yields Theorem A (for X arbitrary) with $C(X, R)$ and $C(X_i, R)$ replaced, respectively, by the lattices $C^*(X, R)$ and $C^*(X_i, R)$ of *bounded* real-valued continuous functions on X and X_i.

[2] Our proof is a modification of Kaplansky's original argument. No separation properties are required of X.

[2, p. 621]): If L_1 and L_2 are lattices and if P is a prime ideal of the direct product $L_1 \times L_2$, then either $P = P_1 \times L_2$ for some prime ideal P_1 of L_1 or $P = L_1 \times P_2$ for some prime ideal P_2 of L_2.

If Y is a subset of X and if $f \in C(X, K)$, then $f | Y$ denotes the restriction of f to Y. If L is a sublattice of $C(X, K)$, then we set

$$L_Y = \{f | Y : f \in L\}.$$

It is clear that L_Y is a sublattice of $C(Y, K)$.

We can now state the following result:

THEOREM B. *Let X be a topological space, let K be a chain equipped with its order topology, and let L be an adequate sublattice of $C(X, K)$. If L is isomorphic to the direct product of two lattices L_1 and L_2, then X is the union of disjoint open-and-closed subsets X_1 and X_2 having the property that L_i is isomorphic to L_{X_i} $(i = 1, 2)$. (The isomorphisms involved are described explicitly below.) Moreover, X_i is nonempty if and only if L_i has at least two distinct elements.*

PROOF. If $x \in X$ and $f \in L$, we set

$$P_x(f) = \{g \in L : g(x) \leqq f(x)\}$$

and

$$P^x(f) = \{g \in L : g(x) \geqq f(x)\}.$$

It is clear that $P_x(f)$ (resp. $P^x(f)$) is a prime (resp. dual prime) ideal of L provided only that it is a proper subset of L. The adequacy of L then ensures that, in any event, either $P_x(f)$ is a prime ideal of L or $P^x(f)$ is a dual prime ideal of L.

We choose now an isomorphism δ from L onto $L_1 \times L_2$ and a fixed element $k \in L$. Denote by $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) the collection of all prime ideals P of L such that $\delta(P)$ is of the form $P_1 \times L_2$ (resp. $L_1 \times P_2$), with P_i a prime ideal of L_i. For $i = 1, 2$, denote by X_i the set of all points $x \in X$ such that either $P_x(k) \in \mathcal{P}_i$ or $L - P^x(k) \in \mathcal{P}_i$. Then it is easily seen that X_1 and X_2 are disjoint and that $X = X_1 \cup X_2$. Moreover, if y is in the closure of X_i, then

$$\bigcap \{P_x(k) : x \in X_i\} \subseteq P_y(k)$$

and

$$\bigcap \{P^x(k) : x \in X_i\} \subseteq P^y(k),$$

from which it follows that $y \in X_i$. Thus both X_1 and X_2 are open-and-closed.

Now let π_i be the projection of $L_1 \times L_2$ onto L_i, and consider the mapping $\phi_i = \pi_i \circ \delta$ from L onto L_i. Let f, $g \in L$ and suppose that $\phi_1(f) \leqq \phi_1(g)$ but that $f(x) > g(x)$ for some $x \in X_1$. Then $P_x(g)$ is a prime ideal of L that contains g but not f. If $P_x(k) \neq L$, then, since $P_x(g) \cap P_x(k)$ contains a prime ideal of L (namely, $P_x(g \wedge k)$), $P_x(g)$ must map onto $P_1 \times L_2$ for some prime ideal P_1 of L_1. But then $\phi_1(f) \in P_1$ so that $f \in P_x(g)$, a contradiction. Moreover, if $P^x(k) \neq L$, then a dual argument again yields a contradiction. Arguing similarly for X_2, we therefore conclude that

$$\phi_i(f) \leqq \phi_i(g) \text{ implies } f \mid X_i \leqq g \mid X_i \qquad (i = 1, 2). \tag{1}$$

Now suppose, on the other hand, that $f|X_1 \leqq g|X_1$ but that $\phi_1(f) \nleqq \phi_1(g)$. Since L_1 is distributive, Zorn's lemma provides a prime ideal P_1 in L_1 that contains $\phi_1(g)$ but not $\phi_1(f)$. Let P be the prime ideal in L that maps onto $P_1 \times L_2$. Then $g \in P$ and $f \notin P$. Let $h = \delta^{-1}(\phi_1(f), \phi_2(g))$ so that $h \notin P$. Now $\phi_2(h) = \phi_2(g)$ and therefore, by (1), $h|X_2 = g|X_2$. But then $f \wedge h \leqq g$ so that $f \wedge h \in P$, a contradiction. Using a similar argument for ϕ_2, we thus obtain

$$f \mid X_i \leqq g \mid X_i \text{ implies } \phi_i(f) \leqq \phi_i(g) \qquad (i = 1, 2). \tag{2}$$

We conclude from (2) that $\psi_i : f|X_i \to \phi_i(f)$ is a well-defined order-preserving map from L_{X_i} onto L_i. Moreover, by (1), ψ_i is one-to-one and ψ_i^{-1} is also order-preserving. Hence ψ_i is an isomorphism.

Using the adequacy of L, note finally that X_i is nonempty if and only if L_{X_i} has at least two distinct elements. Since L_i is isomorphic to L_{X_i}, the last assertion of the theorem is immediate, and the proof is complete.

Remark 1. Let δ and π_i be as above and let λ_i be an arbitrary isomorphism from L_i into $L_1 \times L_2$ such that $\pi_i \circ \lambda_i$ is the identity on L_i. Let ρ_i be the restriction homomorphism $f \to f|X_i$ from L onto L_{X_i}. The proof of Theorem B shows that $\rho_i \circ \delta^{-1} \circ \lambda_i$ is an isomorphism from L_i onto L_{X_i} and that, for each $f \in L$,

$$(\rho_i \circ \delta^{-1} \circ \lambda_i)(\pi_i(\delta(f))) = \rho_i(f) \qquad (i = 1, 2).$$

Remark 2. If $P_x(k)$ is always a prime ideal of L (and this is the case, for example, if K has no last element and if L contains every constant function on X to K), then the proof of Theorem B admits the following simplification: Ignoring $P^x(k)$, we can take X_i to be the set of all $x \in X$ such that $P_x(k) \in \mathcal{P}_i$ (cf. the proof of Theorem 2 of [2]).

In the following corollary, $C^*(X, K)$ denotes the sublattice of $C(X, K)$ consisting of all bounded continuous functions from X to K.

Corollary. *Let X and K be as before. If $C(X, K)$ (resp. $C^*(X, K)$) is isomorphic to the direct product of two lattices L_1 and L_2, then X is the* *union of disjoint open-and-closed subsets X_1 and X_2 having the property that L_i is isomorphic to $C(X_i, K)$ (resp. $C^*(X_i, K)$) $(i=1, 2)$.*

PROOF. If K consists of a single element, we can take $X_1=X$ and $X_2=\varnothing$; the result is then a consequence of the fact that $C(\varnothing, K) =\{\varnothing\}$. If K has at least two elements, then both $C(X, K)$ and $C^*(X, K)$ are adequate, and the result follows immediately from the theorem.

REMARK 3. The following question remains open: What are necessary and sufficient conditions on a sublattice L of $C(X, K)$ in order that a direct decomposition of L be reflected in a corresponding decomposition of X? In any case, the hypothesis of adequacy cannot simply be deleted. To see this, let R be the chain of real numbers, let $i: R\rightarrow R$ be the identity mapping, and let L be the (nonadequate) sublattice of $C(R, R)$ generated by i and $-i$. If K is any chain with exactly two elements, then L is isomorphic to $K\times K$, but there is no corresponding decomposition of R.

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NEW YORK UNIVERSITY