TEST SPACES FOR NORMAL SPACES

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Let Q be a class of topological spaces. For X in Q , let $\dim X$ denote the dimension of X defined in terms of finite open coverings. If Q is the class of all normal spaces, then the dimension of spaces X in Q is characterized by the following theorem [2, p. 212]: $\dim X \leq n$ if, and only if, given any closed subset C of X , any continuous mapping of C into the n -sphere S^n has a continuous extension over X with respect to S^n . The possibility of replacing S^n in the above theorem by a more general space Y was explored in two earlier papers [8; 9]. Such a space Y we call a *test space for Q -spaces of dimension n* . To be precise, Y is a test space for Q -spaces of dimension n if it satisfies the statement: A Q -space X has dimension $\leq n$ if, and only if, given any closed subset C of X , any continuous mapping of C into Y has a continuous extension over X with respect to Y .

A space Y is called an *absolute neighborhood retract relative to the class Q* (abbreviated $\text{ANR}(Q)$) if (a) Y is in Q , and (b) whenever Y is imbedded topologically as a closed subset of a Q -space Z , then Y is a retract of some neighborhood of Y in Z (see [3]). Kodama has shown in [5] that an n -dimensional metric space Y is an $\text{ANR}(\text{metric})$ if and only if Y is LC^n .

The results of [8; 9] may be formulated as follows: if Q is the class of all metric spaces, then an n -dimensional, n -LC, metric space Y is a test space for Q -spaces of dimension n if, and only if, Y is an $\text{ANR}(\text{metric})$ and is the homotopy type of a wedge of n -spheres (i.e., a union of n -spheres with a single common point). In this paper we shall prove a similar theorem for Q the class of all normal spaces.

In [9] we showed that if Y and Z are both $\text{ANR}(\text{metric})$ and of the same homotopy type, then Y is a test space for metric spaces of dimension n if and only if Z is. Needed in the proof was Borsuk's homotopy extension theorem. Dowker showed in [2, p. 205] that the homotopy extension theorem holds for mappings of a normal space into an $\text{ANR}(\text{compact metric})$ provided homotopy is replaced by uniform homotopy. Using this result, we can prove the following lemma:

LEMMA 1. *Let Y and Z be $\text{ANR}(\text{compact metric})$ having the same homotopy type. Then Y is a test space for normal spaces of dimension n if and only if Z is.*

Received by the editors April 24, 1961.

PROOF. Let us use Kuratowski's notation $X\tau Y$ [7, p. 251] to mean that if C is a closed subset of X , then any continuous mapping of C into Y has a continuous extension over X .

Suppose that Z is a test space for normal spaces of dimension n ; we shall show that Y is also. We first show that Y satisfies the condition: $X\tau Y$ implies $\dim X \leq n$ for any normal space X . To do this, let X be any normal space satisfying $X\tau Y$, C any closed subset of X , and $f: C \rightarrow Y$ any continuous mapping. Since Y and Z have the same homotopy type, there exist continuous mappings $\alpha: Y \rightarrow Z$ and $\beta: Z \rightarrow Y$ such that $\alpha\beta$ and $\beta\alpha$ are homotopic to the identity mappings of Z and Y respectively. Since Y and Z are compact metric, it follows from another result of Dowker [2, p. 227] that $\alpha\beta$ and $\beta\alpha$ are *uniformly* homotopic to these identity mappings. Consider the composition mapping $\beta f: C \rightarrow Y$. Since we are given $X\tau Y$, βf has a continuous extension $F: X \rightarrow Y$, and this provides a mapping $\alpha F: X \rightarrow Z$. Now $\alpha F|_C = \alpha\beta f$, and since $\alpha\beta$ is uniformly homotopic to the identity mapping of Z , it follows that $\alpha\beta f$ is uniformly homotopic to f . Dowker's version of Borsuk's homotopy extension theorem now yields a continuous extension of f over X . Therefore $\dim X \leq n$, and the first part of the theorem is proved.

To complete the proof that Y is a test space for normal spaces of dimension n , it remains to be shown that $\dim X \leq n$ implies $X\tau Y$ for any normal space X . This proof is quite similar to the one above, and will not be given.

As in [10, p. 209] we denote by $X \vee Y$ the "wedge product" of the spaces X and Y , that is, the one-point union of X and Y . If $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ are mappings which agree on the common point, then a mapping $f \vee g: X \vee Y \rightarrow Z$ is defined by letting $f \vee g$ be equal to f on X and equal to g on Y . The wedge product will be useful in the following lemma and theorems.

LEMMA 2. *A finite wedge of n -spheres is a test space for normal spaces of dimension n .*

PROOF. This follows as in [8; 9] since the n -sphere tests the dimension of arbitrary normal spaces [2, p. 212].

LEMMA 3. *If the compact metric space Y is a test space for compact metric spaces of dimension n , then*

$$\pi_i(Y) = 0 \text{ for } i < n, \text{ and } Y \text{ is } LC^{n-1}.$$

PROOF. This is clearly a corollary to a theorem of Kuratowski [7, p. 266] who proved that the conclusion is equivalent to the prop-

erty: $\dim X \leq n$ implies $X \tau Y$ for a separable metric space Y . A test space for compact metric spaces, of course, has this property.

We can now prove the main results of this paper.

THEOREM 1. *Let Y be a compact, n -LC, n -dimensional metric space. If Y is a test space for normal spaces of dimension n , then Y is an ANR(metric) and has the homotopy type of a finite wedge of n -spheres*

PROOF. The case $n=1$ requires separate treatment because of the non-abelian character of $\pi_1(Y)$. By hypothesis, Y is a compact 1-dimensional, 1-LC metric space, so by Lemma 3 and Kodama's result [5] Y is an ANR(metric). It follows that Y is locally connected and locally simply connected, so by a result of Curtis and Fort [1, p. 143] $\pi_1(Y)$ is finitely generated and free. Therefore $\pi_1(Y)$ is isomorphic to the fundamental group of finite wedge of 1-spheres. Let $f_i: S^1 \rightarrow Y$ be a finite set of base-point preserving mappings which generate $\pi_1(Y)$. Then it is clear that $\bigvee f_i: VS^1 \rightarrow Y$ induces an isomorphism of $\pi_1(VS^1)$ onto $\pi_1(Y)$. Now VS^1 and Y are both dominated by 1-dimensional CW complexes [4, p. 365] so the "classical theorem" of Whitehead [11, Theorem 3, p. 216] shows that $\bigvee f_i$ is a homotopy equivalence. Therefore VS^1 and Y have the same homotopy type and this completes the proof in the case $n=1$.

Suppose now that $n \geq 2$. Then the hypotheses imply that Y is an n -dimensional, LC n metric space, so by Kodama's theorem, Y is an ANR(metric). Thus [6] the singular and Čech homology groups are isomorphic in each dimension, and since Y is compact, they are finitely generated. Since Y is n -dimensional, $H_i(Y) = 0$ for $i > n$ and $H_n(Y)$ is torsion-free. By Lemma 3 and the Hurewicz Isomorphism Theorem we have $H_n(Y) \approx \pi_n(Y)$ and $H_i(Y) = 0$ for $i < n$; hence $H_n(Y)$ is the only nonzero homology group of Y . Let $f_i: S^n \rightarrow Y$ be a set of base-point preserving mappings which generate $\pi_n(Y)$. Then $\bigvee f_i: VS^n \rightarrow Y$ induces an isomorphism of $\pi_n(VS^n)$ onto $\pi_n(Y)$ and therefore of $H_n(VS^n)$ on $H_n(Y)$. It now follows from the Whitehead theorem that $\bigvee f_i$ is a homotopy equivalence. Therefore Y has the homotopy type of VS^n and this completes the proof of the theorem.

THEOREM 2. *If Y is an ANR(compact metric) and has the homotopy type of a finite wedge of n -spheres, then Y is a test space for normal spaces of dimension n .*

PROOF. By Lemma 2, a finite wedge of n -spheres is a test space for normal spaces of dimension n . Since a finite wedge of n -spheres is an ANR(compact metric), it follows from Lemma 1 that Y is also a test space for normal spaces of dimension n .

Note that if Y is a test space for a certain class of spaces, it is also a test space for any subclass. Thus, a test space for normal spaces of dimension n is also a test space for compact metric spaces of dimension n . It is likewise clear that a compact ANR(metric) is an ANR(compact metric). Therefore Theorems 1 and 2 imply

THEOREM 3. *Let Y be a compact, n -dimensional, n -LC metric space. Then Y is a test space for normal spaces of dimension n if and only if it is an ANR(compact metric) and has the homotopy type of a finite wedge of n -spheres.*

In closing, the author would like to acknowledge some suggestions of the referee which led to a strengthening of the final result.

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