SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

ON THE MEAN-VALUE PROPERTY OF HARMONIC FUNCTIONS

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THEOREM. *Let D be a simply connected plane domain of finite area and t a point of D such that, for every function u harmonic in D and integrable over D , the mean value of u over the area of D equals $u(t)$. Then D is a disc and t is its center.*

PROOF. Let $f(z)$ be any function analytic in D with finite quadratic integral $\iint_D |f(z)|^2 dx dy$; by the Schwarz inequality, the real and imaginary parts of $f(z)$ are each (absolutely) integrable over D , and so

$$(1) \quad f(t) = \iint_D f(z) \cdot A^{-1} dx dy,$$

where A denotes the area of D . On the other hand, for every analytic quadratically integrable function the equality

$$(2) \quad f(t) = \iint_D f(z) (Kz, t)^* dx dy$$

holds, where $K(z, t)$ denotes the "reproducing kernel" [1] of D with parameter-point t . Since $K(z, t)$ is uniquely determined by its reproducing property, it follows that

$$(3) \quad K(z, t) = A^{-1}$$

and hence that

$$(4) \quad \int_t^z K(\zeta, t) d\zeta = A^{-1}(z - t).$$

On the other hand, it is known that the function defined by the left side of (4) maps D onto a disc with center at the origin. (It is at this point in the argument that the simple connectivity of D is employed.) It is now evident from the right side of (4) that D is itself a disc, with center at t .

REFERENCE

1. S. Bergman, *The kernel function and conformal mapping*, Math. Surveys No. 5. Amer. Math. Soc., Providence, R. I., 1950.

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