

ON THE CONVERGENCE OF INFINITE EXPONENTIALS

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Given a sequence of complex numbers $\{a_i\}$, we define a sequence of functions:

$$a_1^z, \quad a_1^{(a_2)^z}, \quad a_1^{(a_2^{(a_3)^z})}, \quad \dots$$

This sequence is formally represented by $E(a_1, a_2, a_3, \dots; z)$, and is called an infinite exponential. If all the $a_i = a$, as is the case in this paper, the infinite exponential is represented by $E(a; z)$. This nomenclature follows Barrow [1]. Another symbolism has been used by Thron [9]. It is necessary to specify the values $\log a_n$ to make the sequence determinate.

Euler [4] was the first to investigate seriously the convergence of the sequence $E(a; z)$. He stated and demonstrated the results for the case where a and z are real. However, his demonstration was not a rigorous proof. Later proofs of these results were given independently by Seidel [8], Gravé [5], and Barrow [1]. We quote the result that $E(a; 1)$ converges when

$$e^{-e} \leq a \leq e^{1/e}.$$

Convergence for complex a_i has been considered by Thron [9]. He showed that $E(a_1, a_2, a_3, \dots; 1)$ will converge if $|\log a_i| \leq e^{-1}$, $i = 1, 2, 3, \dots$. The purpose of this paper is to establish the existence of a region of convergence extending outside this one in the special case that $a_i = a$. If $E(a; z)$ converges to the limit t , then clearly

$$a^t = t \quad \text{or} \quad a = t^{1/t}.$$

If now one lets $t = e^\zeta$ where ζ is a complex parameter, then [2]

$$(1) \quad a = e^{\zeta e^{-\zeta}}.$$

In the remainder of this paper, the representation (1) will be employed.

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$E(a; z)$ is the sequence of iterates of the function $\phi(z)$. That is, it is identical with the sequence $\phi_n(z)$ defined by

$$\begin{aligned}\phi_1(z) &= \phi(z) = a^z, \\ \phi_2(z) &= \phi(\phi_1(z)) = a^{\phi_1(z)}, \\ \phi_n(z) &= \phi(\phi_{n-1}(z)) = a^{\phi_{n-1}(z)}.\end{aligned}$$

Now $\phi(z)$ is an entire function with the fixed point $z_0 = e^{\zeta}$ and the theory of iteration [6; 7, pp. 229–239] tells us that the sequence $\phi_n(z)$ will converge for z in some neighborhood of the fixed point provided that $|\phi'(z_0)| < 1$.

Now using (1)

$$\frac{d\phi}{dz} = \frac{d}{dz} e^{\zeta e^{-\zeta z}} = e^{\zeta e^{-\zeta z}} \zeta e^{-\zeta z}, \quad \frac{d\phi(e^{\zeta})}{dz} = \zeta.$$

Hence we have [2]

THEOREM 1. *If $|\zeta| < 1$ and $a = e^{\zeta e^{-\zeta}}$, then $E(a; z)$ converges to e^{ζ} for z in some neighborhood of $z_0 = e^{\zeta}$.*

The results of this paper are based on the estimate of the convergence neighborhood given in

THEOREM 2. *If $|\zeta| < 1$ and $a = e^{\zeta e^{-\zeta}}$ then $E(a; z)$ converges to e^{ζ} provided $|z - e^{\zeta}| < r_0 |e^{\zeta}|$ where r_0 is the positive root of the equation $r^{-1} \log(1+r) = |\zeta|$.*

The convergence is uniform in

$$(2) \quad |z - e^{\zeta}| \leq \theta r_0 |e^{\zeta}|; \quad 0 < \theta < 1.$$

From the point of view of the theory of iteration of functions $E(a; 1)$ is merely a special case of $E(a; z)$. This was recognized by Euler, but as Condorcet [3] originally raised the question of the convergence of $E(a; 1)$ there is particular interest in determining the values of a for which $E(a; 1)$ converges. We shall deduce from Theorem 2

THEOREM 3. *If $a = e^{\zeta e^{-\zeta}}$ and $|\zeta| \leq \log 2$ then $E(a; 1)$ converges to e^{ζ} .*

A slightly more extensive region than $|\zeta| \leq \log 2$ will be found as stated below in Theorem 4.

The regions of convergence of $E(a; 1)$ in the upper half of the a -plane as established by Thron, Theorem 3 and Theorem 4 are illustrated in Figure I which is based on computations performed on an electronic computer.

Theorem 1 was given by Carlsson [2].

Before proceeding to prove Theorem 2 it is helpful to consider the function

$$\theta(r) = \frac{e^{rs} - 1}{r} = s + \frac{rs^2}{2!} + \frac{r^2s^3}{3!} + \dots$$

We evidently have the following

LEMMA. *If $0 < s < 1$ then $\theta(r)$ is monotone increasing with r for $r > 0$. There exists a unique positive solution $r = r_0$ of the equation $r^{-1} \log(1 + r) = s$ and $\theta(r_0) = 1$. $s < \theta(r) < 1$ for $0 < r < r_0$.*

Now let $z = e^t(1 + \beta)$. We find that $a^z = \exp(z\zeta e^{-t}) = e^t e^{\beta t}$. We propose to establish conditions under which $|a^z - e^t| < |z - e^t|$. Evidently $|a^z - e^t| = |e^t| |e^{\beta t} - 1| \leq |e^t| (e^{|\beta t|} - 1)$. Now with $s = |\zeta|$ we find that provided $|\beta| \leq r$ then

$$|a^z - e^t| \leq |e^t| \theta(r) |\beta| = \theta(r) |z - e^t|.$$

Hence if $|\beta| \leq r < r_0$ we have

$$|\phi_1(z) - e^t| = |\beta_1 e^t| \leq \theta(r) |z - e^t| = \theta(r) |\beta e^t|.$$

This implies that $|\beta_1| \leq \theta(r) |\beta| \leq \theta(r)r \leq r$ and hence repeating the argument

$$|\phi_2(z) - e^t| = |\beta_2 e^t| \leq \theta(r) |\phi_1(z) - e^t| \leq [\theta(r)]^2 |z - e^t|.$$

Evidently $|\phi_n(z) - e^t| \leq [\theta(r)]^n |z - e^t|$ and $\phi_n(z)$ converges uniformly to e^t for $|z - e^t| \leq r |e^t|$. This proves Theorem 2.

We prove Theorem 3 by showing that $z = 1$ is within the circle of convergence associated with each number a in the relevant region. Substituting $|\zeta| = \log 2$ in equation (2) we find $r_0 = 1$. Hence, we shall have convergence in this case if $|\beta| < 1$. The point under consideration is $z = 1 = e^t(1 + \beta)$. Here $\beta = e^{-t} - 1$ and $|\beta| = |e^{-t} - 1| \leq e^{t|s|} - 1 \leq e^{\log 2} - 1 = 1$. Now the equality holds only when $-\zeta = |\zeta| = \log 2$. Otherwise, $|\beta| < 1$. But when $\zeta = -\log 2$, we appeal to the results for real a , which indicate convergence when $a = 1/4$ (corresponding to $\zeta = -\log 2$). This proves Theorem 3.

It is clear that equation (2) assures convergence in a somewhat larger region. Thus letting

$$r = |\beta| = |e^{-t} - 1|$$

we define a contour in the ζ -plane, namely

$$(3) \quad |\zeta| = \frac{\log(1 + |e^{-t} - 1|)}{|e^{-t} - 1|}$$

within which $|e^{\xi} - 1| < r_0 |e^{\xi}|$. Hence, we have

THEOREM 4. *The infinite exponential $E(a; 1)$ converges to e^{ξ} when ξ is within the contour defined by equation (3) and $a = e^{\xi} e^{-\xi}$.*

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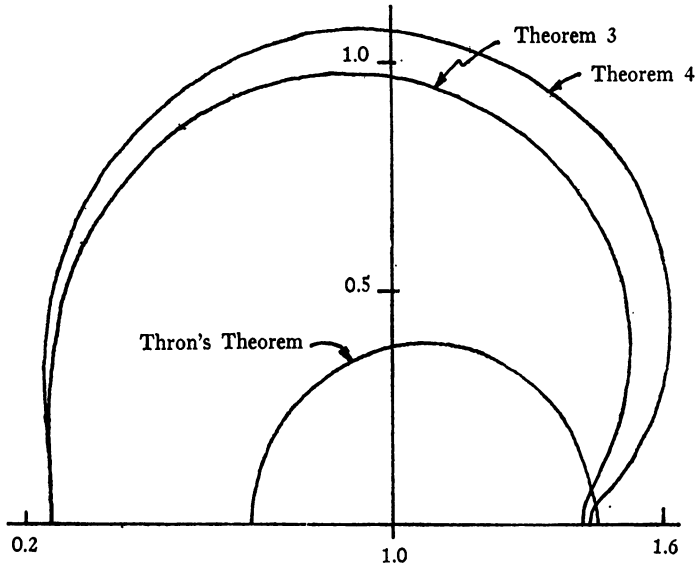


FIGURE 1. Regions of convergence of $E(a; 1)$.

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