A NOTE ON FUNCTIONS OF SEVERAL COMPLEX VARIABLES¹

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In the paper [1] (see also [2]) the following theorem was proved:
The monomials are the only entire functions whose absolute value is constant on the distinguished boundary of the unit polycylinder.

The aim of the note is to give a new proof of this theorem and a slight generalization of it.

Let $\psi_i(z_i)$, $i=1, 2, \dots, n$, be a polynomial of degree $k_i \geq 1$. Let lemniscate

$$(1) \quad \Gamma_i = \{z_i: |\psi_i(z_i)| = 1\}, \quad i = 1, 2, \dots, n$$

be a Jordan curve. Denote by D_i the interior of Γ_i and by D the Cartesian product of D_i , $D = D_1 \times D_2 \times \dots \times D_n$.

THEOREM. *If $f(z_1, \dots, z_n)$ is an entire function such that*

$$(2) \quad |f(z_1, \dots, z_n)| = 1 \text{ for } (z_1, \dots, z_n) \in \Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n,$$

then f may be written in the form

$$(3) \quad f(z_1, \dots, z_n) = e^{i\alpha} \psi_1(z_1)^{m_1} \psi_2(z_2)^{m_2} \dots \psi_n(z_n)^{m_n},$$

where α is a real constant and m_i ($i=1, 2, \dots, n$) is a nonnegative rational number. If $\psi_i(z_i)$ has no multiple zeros then m_i is a nonnegative integer.²

PROOF. 1°. At first we shall prove the theorem under the additional assumption that $\psi_i(z_i) = z_i$, i.e., in the case when D is a unit polycylinder. Our proof will be inductive.

Let $f(z_1)$ be an entire function such that $|f(z_1)| = 1$ for $|z_1| = 1$. Let $a_1, a_2, \dots, a_m$ be zeros (multiple zeros being repeated) of $f(z_1)$ contained in the circle $|z_1| < 1$. Then the function

$$(4) \quad \phi(z_1) = f(z_1) \cdot \frac{(1 - \bar{a}_1 z_1) \dots (1 - \bar{a}_m z_1)}{(z_1 - a_1) \dots (z_1 - a_m)}$$

has no zeros in $|z_1| \leq 1$, and $|\phi(z_1)| = 1$ for $|z_1| = 1$. Therefore, $\phi(z_1) = e^{-i\alpha}$, $\alpha = \text{real const.}$ Hence,

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² The original formulation of the theorem stated that the m_i were always integers. The author is indebted to the referee for the remark that if the zeros of $\psi_i(z_i)$ are not simple, the m_i do not have to be integers.

$$f(z_1) = e^{ia} \frac{(z_1 - a_1) \cdots (z_1 - a_m)}{(1 - \bar{a}_1 z_1) \cdots (1 - \bar{a}_m z_1)}.$$

Since $f(z_1)$ is entire, we have $a_i = 0$, $i = 1, 2, \dots, m$, so $f(z_1) = e^{ia} z_1^m$.

Assume that the theorem (in the case of polycylinder) is true for n variables. We shall prove that it is true for $n+1$ variables.

Let $f(z_1, \dots, z_n, z_{n+1})$ be an entire function such that

$$(5) \quad |f(z_1, \dots, z_n, z_{n+1})| = 1 \text{ for } |z_i| = 1, \quad i = 1, 2, \dots, n+1.$$

The function $f(z_1, \dots, z_{n+1})$ may be written in the form

$$(6) \quad f(z_1, \dots, z_n, z_{n+1}) = \sum_{\mu=0}^{\infty} f_{\mu}(z_1, \dots, z_n) z_{n+1}^{\mu},$$

where $f_{\mu}(z_1, \dots, z_n)$ are entire functions and the series is uniformly absolutely convergent for all $|z_i| < \infty$, $i = 1, 2, \dots, n+1$. By the first induction step we have $f(e^{i\phi_1}, \dots, e^{i\phi_n}, z_{n+1}) = f_{\mu_0}(e^{i\phi_1}, \dots, e^{i\phi_n}) z_{n+1}^{\mu_0}$, where μ_0 depends on $(\phi_1, \phi_2, \dots, \phi_n)$, $\phi_i = \text{real number}$. Let $\phi_1, \dots, \phi_{n-1}$ be fixed for a moment. For any real value of ϕ_n there is a nonnegative integer μ such that $f(e^{i\phi_1}, \dots, e^{i\phi_{n-1}}, e^{i\phi_n}, z_{n+1}) = f_{\mu}(e^{i\phi_1}, \dots, e^{i\phi_{n-1}}, e^{i\phi_n}) z_{n+1}^{\mu}$. Since the set of all points $z_n = e^{i\phi_n}$, $\phi_n = \text{real number}$, is nondenumerable, there exists an integer μ_n and a sequence of different points $z_n^{(\nu)} = e^{i\phi_n^{(\nu)}}$ such that

$$f(e^{i\phi_1}, \dots, e^{i\phi_{n-1}}, e^{i\phi_n^{(\nu)}}, z_{n+1}) = f_{\mu_n}(e^{i\phi_1}, \dots, e^{i\phi_{n-1}}, e^{i\phi_n^{(\nu)}}) z_{n+1}^{\mu_n},$$

$$\nu = 1, 2, \dots$$

Therefore

$$f(e^{i\phi_1}, \dots, e^{i\phi_{n-2}}, z_n, z_{n+1}) \equiv f_{\mu_n}(e^{i\phi_1}, \dots, e^{i\phi_{n-1}}, z_n) z_{n+1}^{\mu_n},$$

z_n, z_{n+1} being arbitrary. Here the integer μ_n depends on $(\phi_1, \dots, \phi_{n-1})$. Letting $\phi_1, \dots, \phi_{n-2}$ be constant, we shall prove in an analogous way that there is an integer $\mu_{n-1} \geq 0$ such that

$$f(e^{i\phi_1}, \dots, e^{i\phi_{n-2}}, z_{n-1}, z_n, z_{n+1}) \equiv f_{\mu_{n-1}}(e^{i\phi_1}, \dots, e^{i\phi_{n-2}}, z_{n-1}, z_n) z_{n+1}^{\mu_{n-1}},$$

z_{n-1}, z_n, z_{n+1} being arbitrary. Repeating this reasoning, we shall finally attain the identity $f(z_1, \dots, z_{n+1}) \equiv f_{\mu_0}(z_1, \dots, z_n) z_{n+1}^{\mu_0}$, where μ_0 is a nonnegative integer and $f_{\mu_0}(z_1, \dots, z_n)$ is an entire function. Due to (5) we have

$$|f(z_1, \dots, z_{n+1})| = |f_{\mu_0}(z_1, \dots, z_n)| = 1 \text{ for } |z_i| = 1,$$

$$i = 1, 2, \dots, n+1.$$

Therefore, by the induction assumption

$$f_{\mu_0}(z_1, \dots, z_n) = e^{i\alpha \frac{m_1}{z_1} \frac{m_2}{z_2} \dots \frac{m_n}{z_n}},$$

whence

$$(7) \quad f(z_1, \dots, z_n, z_{n+1}) \equiv e^{i\alpha \frac{m_1}{z_1} \dots \frac{m_n}{z_n} \frac{m_{n+1}}{z_{n+1}}} \quad (m_{n+1} = \mu_0),$$

and the proof is completed.

2°. To prove our theorem in the general case, let us observe that for any $i = 1, 2, \dots, n$ there is a univalent branch of

$$w_i = (\psi_i(z_i))^{1/k_i} \quad \text{for } z_i \in D_i.$$

Indeed, we have $\psi_i(z_i) \neq 0$ for $z_i \in D_i$ (see [3, p. 55]). By the theorem of monodromy there exists a single valued branch $\phi_i(z_i)$ of $(\psi_i(z_i))^{1/k_i}$ in $\mathcal{C}D_i$. We have $1/|\Phi_i(z_i)| < 1$ for $z_i \in \mathcal{C}D_i$ and $1/|\Phi_i(z_i)| = 1$ on the boundary of $\mathcal{C}D_i$. Let w_i be an arbitrary complex number such that $|w_i| < 1$. We have $1/|\Phi_i(z_i)| > |w_i|$ on the boundary of $\mathcal{C}D_i$. Therefore by the theorem of Rouché the functions $1/\Phi_i(z_i)$ and $1/\Phi_i(z_i) - w_i$ have the same number of zeros in $\mathcal{C}D_i$. Since the function $1/\Phi_i(z_i)$ has exactly one zero (at $z_i = \infty$), therefore $w_i = 1/\Phi_i(z_i)$ is a one to one mapping of $\mathcal{C}D_i$ onto $|w_i| < 1$. Let $z_i = \phi(w_i)$ be the inverse function of $\Phi_i(z_i)$. Then the function

$$F(w_1, \dots, w_n) = f(\phi_1(w_1), \dots, \phi_n(w_n))$$

is holomorphic for $1 - \epsilon \leq |w_i| < \infty$ ($i = 1, \dots, n$) and $|F(w_1, \dots, w_n)| = 1$ for $|w_i| = 1$. We have

$$F(w_1, \dots, w_n) = \sum_{\mu_1, \dots, \mu_n = -\infty}^{\infty} a_{\mu_1 \dots \mu_n} w_1^{\mu_1} \dots w_n^{\mu_n},$$

the series being uniformly convergent for $1 - \epsilon \leq |w_i| < \infty$. Therefore, there is a $\delta > 0$ such that the series

$$\bar{F}\left(\frac{1}{w_1}, \dots, \frac{1}{w_n}\right) = \sum_{\mu_1, \dots, \mu_n = -\infty}^{\infty} \bar{a}_{\mu_1 \dots \mu_n} \left(\frac{1}{w_1}\right)^{\mu_1} \dots \left(\frac{1}{w_n}\right)^{\mu_n}$$

is absolutely uniformly convergent for $1 - \delta \leq |w_i| \leq 1 + \delta$. So the function $\bar{F}(1/w_1, \dots, 1/w_n)$ is holomorphic for $1 - \delta \leq |w_i| \leq 1 + \delta$. We have $\bar{F}(1/w_1, \dots, 1/w_n) = (F(1/\bar{w}_1, \dots, 1/\bar{w}_n))^-$. Then $\bar{F}(1/w_1, \dots, 1/w_n) = 1/F(w_1, \dots, w_n)$ for $|w_i| = 1$. Hence $\bar{F}(1/w_1, \dots, 1/w_n) = 1/F(w_1, \dots, w_n)$ for $1 - \epsilon \leq |w_i| < \infty$, and therefore $F(w_1, \dots, w_n)$ is analytically continuable to a meromorphic (single valued) function in the set

$$\{w_1, \dots, w_n: 0 < |w_i| < \infty, i = 1, \dots, n\}.$$

(What we just have proven is simply the Riemann-Schwarz reflection principle for n complex variables.) Let us observe that

$$f(z_1, \dots, z_n) = F((\psi_1(z_1))^{1/k_1}, \dots, (\psi_n(z_n))^{1/k_n}).$$

This equality implies that $F(w_1, \dots, w_n)$ has no poles and therefore it is entire. For $z_i \in \Gamma_i$ we have $|w_i| = |(\psi_i(z_i))^{1/k_i}| = 1$, whence $|F(w_1, \dots, w_n)| = |f(z_1, \dots, z_n)| = 1$. Therefore, by 1° we have $F(w_1, \dots, w_n) = e^{i\alpha} w_1^{\mu_1} \dots w_n^{\mu_n}$, and it follows that

$$f(z_1, \dots, z_n) = e^{i\alpha} [(\psi_1(z_1))^{1/k_1}]^{\mu_1} \dots [(\psi_n(z_n))^{1/k_n}]^{\mu_n},$$

$\alpha = \text{real const}$, $\mu_i = \text{nonnegative integer}$. Since $f(z_1, \dots, z_n)$ is a single valued function, the $[(\psi_i(z_i))^{1/k_i}]^{\mu_i}$ ($i = 1, \dots, n$) has to be single valued. Let $\psi_i(z_i) = A_i(z_i - a_1)^{l_1} \dots (z_i - a_s)^{l_s}$, where $l_1 + l_2 + \dots + l_s = k_i$ and $a_j \neq a_k$ for $j \neq k$. Since $[(\psi_i(z_i))^{1/k_i}]^{\mu_i}$ is single valued, then $(z_i - a_r)^{\mu_i l_r / k_i}$ ($r = 1, \dots, s$) has to be single valued, i.e., the integer $\mu_i l_r$ must have the form $\mu_i l_r = k_i n_r$, where n_r is an integer, therefore $m_i = \mu_i / k_i = n_r / l_r$, $r = 1, 2, \dots, s$. If $l_1 = l_2 = \dots = l_s = 1$, then m_i is an integer, Q.E.D.

BIBLIOGRAPHY

1. R. Bojanic and W. Stoll, *A characterization of monomials*, Proc. Amer. Math. Soc. 13 (1962), 115-116.
2. S. Bochner, *Entire functions in several variables with constant absolute values on a circular uniqueness set*, Proc. Amer. Math. Soc. 13 (1962), 117-120.
3. J. L. Walsh, *Interpolation and approximation*, 3rd ed., Boston, Mass., 1960.

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