ON THE CIRCLE METHOD OF SUMMATION OF A CAUCHY PRODUCT SERIES[1]

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1. Concerning the Euler summability of a Cauchy product series Knopp [1; 2] proved the theorems of Abel's and Mertens' type, and later Hara [3] proved the theorem of Cauchy's type. Here the Euler means of a sequence $\{s_n\}$ depend on a parameter r, and are defined by the transform

$$\sigma_n(r) = \sigma_n = \sum_{\nu=0}^{n} \binom{n}{\nu} r^\nu (1-r)^{n-\nu} s_\nu, \qquad n = 0, 1, 2, \cdots.$$

We assume $0<r\leqq 1$, in which case the summation method is regular [2]. The case $r=1$ corresponds to the ordinary convergence. We denote this method of summation by (ϵ, r).

On the other hand the circle means of a sequence $\{s_n\}$ depend on a parameter r and are defined by the transform

$$\overset{*}{\sigma}_n(r) = \overset{*}{\sigma}_n = \sum_{\nu=n}^{\infty} \binom{\nu}{n} r^{n+1} (1-r)^{\nu-n} s_\nu, \qquad n = 0, 1, 2, \cdots.$$

Here we assume $0<r\leqq 1$, in which case the summation is regular also. We denote this method of summation by (γ, r).

The relations between (ϵ, r)-method and (γ, r)-method are studied by Meyer-König [4], Vermes [5; 6], Ramanujan [7; 8] and the author [9].

The purpose of this short note is to show that the theorems of Abel's and Mertens' and Cauchy's type all hold for the (γ, r)-method of summation.

2. When a series $\sum_{n=0}^\infty a_n$ is given, let s_n, $n=0, 1, 2, \cdots$, be the partial sums of this series. If the sequence $\{s_n\}$ is summable by (γ, r)-method to A, then we shall say that

$$\sum_{n=0}^{\infty} a_n = A(\gamma, r).$$

This is the same as

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$$\sum_{n=0}^{\infty} a_n^* = A,$$

where

$$a_n^* = \sum_{\nu=n}^{\infty} \binom{\nu}{n} r^n (1-r)^{\nu-n} a_\nu, \qquad n=0, 1, 2, \cdots \text{ (see [2])}.$$

If $\sum a_n^*$ converges absolutely to A, then we shall say that

$$\sum a_n = A(|\gamma, r|).$$

Furthermore if series $\sum a_n$ and $\sum b_n$ are given, then we shall denote that

$$c_p = \sum_{m+n=p} a_m b_n, \qquad p = 0, 1, 2, \cdots.$$

Here we have the next four theorems.

THEOREM 1. *If* $\sum a_n = A(\gamma, r)$, $\sum b_n = B(\gamma, r)$ *and* $\sum c_n = C(\gamma, r)$, *then*

$$AB = C.$$

THEOREM 2. *If* $\sum a_n = A(|\gamma, r|)$ *and* $\sum b_n = B(\gamma, r)$, *then*

$$\sum c_n = C(\gamma, r), \quad \text{and} \quad C = AB.$$

THEOREM 3. *If* $\sum a_n = A(|\gamma, r|)$ *and* $\sum b_n = B(|\gamma, r|)$, *then*

$$\sum c_n = C(|\gamma, r|), \quad \text{and} \quad C = AB.$$

THEOREM 4. *If* $\sum a_n = A(\gamma, r)$ *and* $\sum b_n = B(\gamma, r)$, *then*

$$\sum c_n = AB(\gamma, r; C, 1)$$

in the sense that the (γ, r) *transform of* $\sum c_n$ *is Cesàro* $(C, 1)$ *summable to* AB.

PROOF. Let

$$f(x) = \sum_{m=0}^{\infty} a_m x^m,$$

and $x = 1 - r + ry$; then

$$f(x) = \sum_{m=0}^{\infty} a_m (1 - r + ry)^m = \sum_{m=0}^{\infty} \frac{f^{(m)}(1-r)}{m!} (ry)^m = \sum_{m=0}^{\infty} a_m^* y^m,$$

since

$$a_m^* = \frac{f^{(m)}(1-r)}{m!} r^m.$$

Similarly we get

$$g(x) = \sum b_n x^n = \sum b_n^* y^n,$$

$$h(x) = \sum c_p x^p = \sum c_p^* y^p,$$

and

$$f(x)g(x) = h(x),$$

for small x and y, (see [2]). Hence

$$\sum_{m+n=p} a_m^* b_n^* = c_p^*.$$

On the other hand for the series $\sum a_m^*$, $\sum b_n^*$ and $\sum c_p^*$ we know the classical Abel, Mertens, Cauchy theorems and so on, (see [2, Theorems 160, 161, 162, 184]).

Thus the proofs of Theorem 1, 2 and 3 are complete.

Professor M. S. Ramanujan advised me to add Theorem 4. I wish to express my hearty thanks to him.

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