Let $r = \epsilon(h' + cj)$, then $r = \epsilon(h') \cdot \epsilon(cj) = r' \epsilon(cj)$, and from (5.2), we have

$$\begin{aligned}\sigma(r) &= \sigma(r') \cdot \sigma(\epsilon(cj)) = \sigma(r') \cdot \epsilon(\sigma(cj)) \\ &= \sigma(r') \epsilon(cj) = sr' \cdot \epsilon(cj) = sr.\end{aligned}$$

Moreover, (6.3) and (6.4) tell us that r is regular in G . Lemma 2 is thus proved.

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SOME INTEGRAL FORMULAS FOR THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST AND SECOND KIND

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In the usual notation let

$$(1) \quad K(k) = \frac{\pi}{2} \sum_0^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!n!} k^{2n}$$

$$(2) \quad E(k) = \frac{\pi}{2} \sum_0^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!n!} k^{2n}$$

denote the complete elliptic integrals of the first and second kind, respectively. It is easily verified that (2) implies

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$$(3) \quad \frac{E(k)}{1-k^2} = \frac{\pi}{2} \sum_0^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{n!n!} k^{2n}.$$

We now consider the integral

$$(4) \quad \int_0^{\pi/2} \frac{E(ik \sin \theta)}{1+k^2 \sin^2 \theta} \frac{E(k \cos \theta)}{1-k^2 \cos^2 \theta} \sin \theta \cos \theta d\theta.$$

Making use of (3) this becomes

$$\begin{aligned} & \frac{\pi^2}{4} \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{1}{2}\right)_r \left(\frac{3}{2}\right)_r}{r!r!} k^{2r} \sum_{s=0}^{\infty} \frac{\left(\frac{1}{2}\right)_s \left(\frac{3}{2}\right)_s}{s!s!} k^{2s} \\ & \quad \cdot \int_0^{\pi/2} \sin^{2r+1} \theta \cos^{2s+1} \theta d\theta \\ &= \frac{\pi^2}{8} \sum_{r,s=0}^{\infty} (-1)^r \frac{\left(\frac{1}{2}\right)_r \left(\frac{3}{2}\right)_r}{r!r!} \frac{\left(\frac{1}{2}\right)_s \left(\frac{3}{2}\right)_s}{s!s!} k^{2r+2s} \frac{r!s!}{(r+s+1)!} \\ (5) \quad &= \frac{\pi^2}{8} \sum_{n=0}^{\infty} \frac{k^{2n}}{(n+1)!} \sum_{r=0}^n (-1)^r \frac{\left(\frac{1}{2}\right)_r \left(\frac{3}{2}\right)_r \left(\frac{1}{2}\right)_{n-r} \left(\frac{3}{2}\right)_{n-r}}{r!(n-r)!} \\ &= \frac{\pi^2}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{n!(n+1)!} k^{2n} \sum_{r=0}^n \frac{(-n)_r \left(\frac{1}{2}\right)_r \left(\frac{3}{2}\right)_r}{r! \left(\frac{1}{2}-n\right)_r \left(-\frac{1}{2}-n\right)_r} \\ &= \frac{\pi^2}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{n!(n+1)!} k^{2n} \cdot {}_3F_2 \left[\begin{matrix} -n, & \frac{1}{2}, & \frac{3}{2}; \\ \frac{1}{2}-n, & -\frac{1}{2}-n \end{matrix} \right]. \end{aligned}$$

Clearly the ${}_3F_2$ vanishes for odd n ; for even n we apply Dixon's theorem [1, p. 13]:

$$(6) \quad {}_3F_2 \left[\begin{matrix} -2n, & \alpha, & \beta; \\ 1-\alpha-2n, & 1-\beta-2n \end{matrix} \right] = \frac{(2n)!(\alpha)_n(\beta)_n(\alpha+\beta)_{2n}}{n!(\alpha+\beta)_n(\alpha)_{2n}(\beta)_{2n}}.$$

In particular

$$\begin{aligned}
{}_3F_2 \left[\begin{matrix} -2n, & \frac{1}{2}, & \frac{3}{2}; \\ \frac{1}{2} - 2n, & -\frac{1}{2} - 2n \end{matrix} \right] &= \frac{(2n)!}{n!} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{(n+1)!} \frac{(2n+1)!}{\left(\frac{1}{2}\right)_{2n} \left(\frac{3}{2}\right)_{2n}} \\
&= \frac{(2n)!}{n!} \frac{(2n+1)!}{(n+1)!} \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{1}{2}\right)_{2n}} \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{3}{2}\right)_{2n}},
\end{aligned}$$

so that (5) becomes

$$\begin{aligned}
\frac{\pi^2}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{2n} \left(\frac{3}{2}\right)_{2n}}{(2n)!(2n+1)!} \frac{(2n)!(2n+1)!}{n!(n+1)!} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_{2n} \left(\frac{3}{2}\right)_{2n}} k^{4n} \\
= \frac{\pi^2}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{n!(n+1)!} k^{4n}.
\end{aligned}$$

Since by (1) and (2)

$$\begin{aligned}
K(k) - E(k) &= \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!n!} - \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!n!} \right] k^{2n} \\
&= \frac{\pi}{2} \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_{n-1}}{n!(n-1)!} k^{2n} \\
&= \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{3}{2}\right)_n}{n!(n+1)!} k^{2n+2},
\end{aligned}$$

it follows that

$$\begin{aligned}
(7) \quad \int_0^{\pi/2} E(ik \sin \theta) E(k \cos \theta) \frac{\sin \theta \cos \theta d\theta}{(1 + k^2 \sin^2 \theta)(1 - k^2 \cos^2 \theta)} \\
= \frac{\pi}{2k^4} (K(k^2) - E(k^2)).
\end{aligned}$$

In the next place we have

$$\begin{aligned}
 & \int_0^{\pi/2} K(ik \sin \theta) K(k \cos \theta) \sin \theta \cos \theta d\theta \\
 &= \frac{\pi^2}{8} \sum_{r,s=0}^{\infty} (-1)^r \frac{\left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_r}{r!r!} \frac{\left(\frac{1}{2}\right)_s \left(\frac{1}{2}\right)_s}{s!s!} \frac{r!s!}{(r+s+1)!} k^{2r+2s} \\
 &= \frac{\pi^2}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!(n+1)!} k^{2n} \sum_{r=0}^n \frac{(-n)_r \left(\frac{1}{2}\right)_r \left(\frac{1}{2}\right)_r}{r! \left(\frac{1}{2}-n\right)_r \left(\frac{1}{2}-n\right)_r}.
 \end{aligned}$$

Applying (6) this reduces to

$$\frac{\pi^2}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!n!} \frac{k^{4n}}{2n+1},$$

so that

$$\begin{aligned}
 (8) \quad \int_0^{\pi/2} K(ik \sin \theta) K(k \cos \theta) \sin \theta \cos \theta d\theta &= \frac{\pi^2}{8} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!n!} \frac{k^{4n}}{2n+1} \\
 &= \frac{\pi}{4k^2} \int_0^k K(x) dx.
 \end{aligned}$$

In exactly the same way we find that

$$\begin{aligned}
 (9) \quad \int_0^{\pi/2} E(ik \sin \theta) E(k \cos \theta) \sin \theta \cos \theta d\theta \\
 = \frac{\pi^2}{16} + \frac{\pi^2}{16} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{n!n!} \frac{k^{4n}}{2n+1} = \frac{\pi^2}{16} + \frac{\pi}{8k^2} \int_0^k E(x) dx.
 \end{aligned}$$

Since

$$\frac{d}{dk} (kE(k)) = 2E(k) - K(k),$$

it follows from (8) and (9) that

$$(10) \quad \int_0^{\pi/2} \{4E(ik \sin \theta)E(k \cos \theta) - K(ik \sin \theta)K(k \cos \theta)\} \sin \theta \cos \theta d\theta \\ = \frac{\pi^2}{4} + \frac{\pi}{4} E(k^2).$$

In exactly the same way one can prove the following more general results. Put

$$A = \int_0^{\pi/2} F[a, b; c; -x \sin^2 \theta] F[a, b; d; x \cos^2 \theta] \sin^{2c-1} \theta \cos^{2d-1} \theta d\theta,$$

$$B = \int_0^{\pi/2} F[a, -a; c; -x \sin^2 \theta] F[a, -a; d; x \cos^2 \theta] \sin^{2c-1} \theta \cos^{2d-1} \theta d\theta.$$

Then

$$A = \frac{\Gamma(c)\Gamma(d)}{2\Gamma(c+d)} {}_4F_3 \left[\begin{matrix} a, b, \frac{1}{2}(a+b), \frac{1}{2}(a+b+1); \\ a+b, \frac{1}{2}(c+d), \frac{1}{2}(c+d+1); \end{matrix} \right. \left. \begin{matrix} x^2 \\ \end{matrix} \right],$$

$$B = \frac{\Gamma(c)\Gamma(d)}{4\Gamma(c+d)} \left\{ 1 + {}_2F_2 \left[\begin{matrix} \frac{1}{2}, a, -a; \\ \frac{1}{2}(c+d), \frac{1}{2}(c+d+1); \end{matrix} \right. \left. \begin{matrix} x^2 \\ \end{matrix} \right] \right\}.$$

If in the evaluation of A we take $a=1/2$, $b=3/2$, $c=d=1$ and replace x by k we get (7); if we take $a=b=1/2$, $c=d=1$ we get (8). Similarly if in the evaluation of B we take $a=1/2$, $c=d=1$ and replace x by k we get (9).

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