OPERATORS ON A QUASI-REFLEXIVE BANACH SPACE

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1. Introduction. Let X be a Banach space, $\mathfrak{B}(X)$ the closure of the algebra of all bounded linear operators on X of finite rank, the closure being taken in the topology of the norm

$$\|B\| = \sup_{x \in X} \frac{\|Bx\|}{\|x\|}, \quad B \in \mathfrak{B}(X).$$

The present paper is concerned with a generalization of a theorem of F. Bonsall and A. W. Goldie [1] which states that if X is reflexive then $\mathfrak{B}(X)$ is an annihilator algebra.

It is shown that if X is quasi-reflexive the algebra $\mathfrak{B}(X)$ can be written as the direct sum of four closed subalgebras

$$\mathfrak{B}(X) = \mathfrak{A}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3,$$

where $\mathfrak{A}_1$ is a right annihilator algebra which is annihilated on the right by the right ideal $\mathfrak{B}_2$ and $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ is a left annihilator algebra which is annihilated on the left by the nilpotent algebra $\mathfrak{B}_3$. Moreover if X is reflexive we have $\mathfrak{B}_1 = \mathfrak{B}_2 = \mathfrak{B}_3 = (0)$, so that the above mentioned theorem is obtained as a special case of the present result.

2. Definitions and notation. Let X be a Banach space, X^* and X^{**} its first and second conjugate spaces. The symbol π will be used to denote the canonical isomorphism of X into X^{**} . The annihilator in X^* of a subspace Y of X will be denoted by Y^+ .

If $x \in X$ and $x^* \in X^*$ then as in [4] we use the symbol $x \otimes x^*$ to denote the one dimensional operator on X defined by the equation

$$(x \otimes x^*)y = x^*(y)x \quad \text{for each } y \in X.$$

Throughout this paper the closure of the algebra of bounded linear operators of finite rank will be denoted by $\mathfrak{B}(X)$. All algebras of operators under consideration are considered to be normed with the operator bound, that is

$$\|A\| = \sup_{x \in X} \frac{\|Ax\|}{\|x\|}.$$

If A is an operator on a Banach space X , we denote by A^* the

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adjoint of A , that is the operator on X^* defined by $A^*x^*(x) = x^*(Ax)$. The symbol I will denote the identity operator.

Let $\mathfrak{A}$ be an algebra and $\mathfrak{E} \subset \mathfrak{A}$. The right (left) annihilator of $\mathfrak{E}$ will be denoted by $R(\mathfrak{E})$ ($L(\mathfrak{E})$).

An algebra $\mathfrak{A}$ is called a *right (left) annihilator algebra* [4] if for every closed left (right) ideal $\mathfrak{J}$ we have $R(\mathfrak{J}) = (0)$, ($L(\mathfrak{J}) = (0)$) if and only if $\mathfrak{J} = \mathfrak{A}$.

A Banach space X is called *quasi-reflexive of order n* [2] if the quotient space $X^{**}/\pi X$ is (finite) n -dimensional.

3. Preliminary lemmas.

3.1. LEMMA. Let X be a Banach space, Y_i , $i = 1, 2$, closed linear subspaces of X such that $X = Y_1 \oplus Y_2$. Let $\mathfrak{B}_i$ be the subset of $\mathfrak{B}(X)$ consisting of those operators in $\mathfrak{B}(X)$ with range contained in Y_i , $i = 1, 2$. Then $\mathfrak{B}_i$ is a closed right ideal in $\mathfrak{B}(X)$ and $\mathfrak{B}(X) = \mathfrak{B}_1 \oplus \mathfrak{B}_2$.

PROOF. Let $A \in \mathfrak{B}_i$, $B \in \mathfrak{B}(X)$, then $\text{Range } AB \subset Y_i$, therefore $\mathfrak{B}_i$ is a right ideal. If $A_n \in \mathfrak{B}_i$ and $A_n \rightarrow A$ in the norm topology of $\mathfrak{B}(X)$, then for each $x \in X$, $A_n x \in Y_i$ and $A_n x \rightarrow Ax$ in the norm topology of X . Since Y_i is closed, $Ax \in Y_i$, so $A \in \mathfrak{B}_i$. Thus, it follows that $\mathfrak{B}_i$ is a closed right ideal in $\mathfrak{B}(X)$. Now, since $X = Y_1 \oplus Y_2$ there exists a continuous projection P of X onto Y_1 with null space Y_2 . If $B \in \mathfrak{B}(X)$ we can write $B = PB + (I - P)B$, where $PB \in \mathfrak{B}_1$ and $(I - P)B \in \mathfrak{B}_2$. This decomposition is obviously unique and therefore $\mathfrak{B}(X) = \mathfrak{B}_1 \oplus \mathfrak{B}_2$.

LEMMA 2. Let X , Y_1 , $\mathfrak{B}_1$ be as in Lemma 1, let Z_i , $i = 1, 2$ be closed linear subspaces of X^* such that $X^* = Z_1 \oplus Z_2$. Let $\mathfrak{A}_i$ be the subset of $\mathfrak{B}_1$ whose elements are the operators in $\mathfrak{B}_1$ whose adjoints have range contained in Z_i , $i = 1, 2$. Then $\mathfrak{A}_i$ is a closed left ideal in $\mathfrak{B}_1$ and $\mathfrak{B}_1 = \mathfrak{A}_1 \oplus \mathfrak{A}_2$.

PROOF. The proof that $\mathfrak{A}_i$ is a closed left ideal is similar to the argument in Lemma 1 and hence is omitted. Let P be the continuous projection of X onto Y_1 with null space Y_2 and Q the continuous projection of X^* onto Z_1 with null space Z_2 . Note first that $\mathfrak{A}_i$, $i = 1, 2$, are closed subspaces of $\mathfrak{B}_1$ and that if $A_i \in \mathfrak{A}_i$, $i = 1, 2$, we have $\|A_1\| = \|Q(A_1 + A_2)\| \leq \|Q\| \|A_1 + A_2\|$. It follows from Theorem 2.1 [3] that $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is a closed subalgebra of $\mathfrak{B}_1$. Next, we show that $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is dense in $\mathfrak{B}_1$. A one dimensional operator in $\mathfrak{B}_1$ is of the form $y \otimes x^*$ with $y \in Y$, $x^* \in X^*$ and can be written $y \otimes x^* = y \otimes Qx^* + y \otimes (I - Q)x^*$, with $y \otimes Qx^* \in \mathfrak{A}_1$ and $y \otimes (I - Q)x^* \in \mathfrak{A}_2$. It follows that every operator of finite rank in $\mathfrak{B}_1$ can be written as the sum of an operator in $\mathfrak{A}_1$ and an operator in $\mathfrak{A}_2$. Now, if $B \in \mathfrak{B}_1 \subset \mathfrak{B}(X)$, there exists a sequence F_n of operators of finite rank on X such that $F_n \rightarrow B$ in

the norm topology. Since P is a continuous projection on Y_1 and $\text{Range } B \subset Y_1$ we also have $PF_n \rightarrow B$ where PF_n is an operator of finite rank belonging to $\mathfrak{B}_1$ and therefore $PF_n \in \mathfrak{A}_1 \oplus \mathfrak{A}_2$. It follows that $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is dense in $\mathfrak{B}_1$ and therefore $\mathfrak{A}_1 \oplus \mathfrak{A}_2 = \mathfrak{B}_1$.

LEMMA 3. *If X is quasi-reflexive of order n then there exists Banach spaces X_1 and X_2 and an equivalent norm for X such that $X_2^* = X_1$, $X_1^* = X$ and*

$$X^* = X_1^{**} = \pi_1 X_1 \oplus (\pi_2 X_2)^+,$$

$$X = X_2^{**} = \pi_2 X_2 \oplus V^+,$$

$$X_1 = X_2^* = V \oplus U,$$

where π_i is the canonical embedding of X_i in X_i^{**} , $i = 1, 2$, and where U , V^+ and $(\pi_2 X_2)^+$ are all (finite) n -dimensional.

PROOF. Since X is quasi-reflexive of order n , it follows from Theorem 3.5 of [2] that X_1, X_2 exist such that $X_2^* = X_1$ and $X_1^* = X$ and such that both X_1 and X_2 are quasi-reflexive of order n . The direct sum decompositions follow at once from Theorem 3.3 and the proof of Theorem 3.1 of [2].

4. Theorem. *Let X be a quasi-reflexive space. Then $\mathfrak{B}(X) = \mathfrak{A}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3$ where $\mathfrak{A}_1$ is a right annihilator algebra and $\mathfrak{B}_2$ a right ideal, which annihilates $\mathfrak{A}_1$ on the right; $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ is a left annihilator algebra and $\mathfrak{B}_3$ a nilpotent algebra which annihilates $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ on the left. Moreover the following are equivalent:*

- (a) $\mathfrak{A}_1$ is a left annihilator algebra,
- (b) $\mathfrak{B}_i = (0)$, $i = 1, 2$, or 3 ,
- (c) X is reflexive.

PROOF. By Lemma 3 there exist Banach spaces X_1 and X_2 such that

$$X^* = \pi_1 V \oplus \pi_1 U \oplus (\pi_2 X_2)^+ \quad \text{and} \quad X = \pi_2 X_2 \oplus V^+.$$

Let $\mathfrak{A}_1$ denote the subalgebra of $\mathfrak{B}(X)$ whose elements are the operators in $\mathfrak{B}(X)$ with range in $\pi_2 X_2$ and which have adjoints with range in $\pi_1 V$; let $\mathfrak{B}_1$ denote the subalgebra of $\mathfrak{B}(X)$ whose elements are those operators in $\mathfrak{B}(X)$ with range in $\pi_2 X_2$, which have adjoints with range in $\pi_1 U$; let $\mathfrak{B}_2$ denote the right ideal of $\mathfrak{B}(X)$ whose elements are the operators in $\mathfrak{B}(X)$ with range in V^+ ; and let $\mathfrak{B}_3$ be the subalgebra of $\mathfrak{B}(X)$ consisting of those operators with range in $\pi_2 X_2$ and whose adjoints have range in $(\pi_2 X_2)^+$. Then an application of Lemma 1 and Lemma 2 yields

$$\mathfrak{B}(X) = \mathfrak{A}_1 \oplus \mathfrak{B}_1 \oplus \mathfrak{B}_2 \oplus \mathfrak{B}_3.$$

Next we will show that $\mathfrak{A}_1$ is a right annihilator algebra. By the proof of Theorem 3.1 [2] we see that there exists an isomorphism α of $\pi_2 X_2$ onto V^* such that $\alpha(\pi_2 x_2)(v) = v(x_2)$ for all $x_2 \in X_2, v \in V$.

We establish next an isomorphism β of $\mathfrak{A}_1$ onto $(\mathfrak{B}(V))^*$.² For $A \in \mathfrak{A}_1$ define $\beta A = \alpha A \alpha^{-1}$. If we let $T \in \mathfrak{B}(V)$ be defined by $Tv = \pi_1^{-1} A^* \pi_1 v$ for all $v \in V$, then

$$(T^* v^*)(v) = v^*(\pi_1^{-1} A^* \pi_1 v) = (A^* \pi_1 v)(\alpha^{-1} v^*) = (\alpha A \alpha^{-1} v^*)(v),$$

and it follows that $\beta A = T^*$ and hence $\beta A \in (\mathfrak{B}(V))^*$.

The mapping β is onto since if $T^* \in (\mathfrak{B}(V))^*$ we can define $A \in \mathfrak{B}(X)$ by $A = \alpha^{-1} T^* \alpha P$, where P is the continuous projection of X onto $\pi_2 X_2$, with null space V^+ ; clearly $\text{Range } A \subset \pi_2 X_2$, and $\beta A = T^* \alpha P \alpha^{-1} = T^*$. To show that $\text{Range } A^* \subset \pi_1 V$ we proceed as follows. Suppose first that T is a one-dimensional operator, then $T = v \otimes v^*$ for some $v \in V, v^* \in V^*$, and $T^* = v^* \otimes \pi v$, where π is the canonical embedding of V into V^{**} . We then have $A = \alpha^{-1}(v^* \otimes \pi v) \alpha P$ and for any $x^* \in X^*, x \in X$ we obtain $A^* x^*(x) = x^*[\alpha^{-1}(v^* \otimes \pi v) \alpha P x] = \pi v(\alpha P x) x^*(\alpha^{-1} v^*) = (P x)(v) x^*(\alpha^{-1} v^*)$ using the definition of α . Now $x = P x + (I - P)x$ and $(I - P)x \in V^+$, so $\pi_1 v(x) = x(v) = P x(v)$, and hence $A^* x^* = x^*(\alpha^{-1} v^*) \pi_1 v \in \pi_1 V$. This shows that $\text{Range } A^* \subset \pi_1 V$ in case T is a one-dimensional operator. Similarly if T is of finite rank we obtain $\text{Range } A^* \subset \pi_1 V$. Finally, if T is arbitrary in $\mathfrak{B}(V)$, there exists a sequence T_n of operators of finite rank such that $T_n \rightarrow T$. Let $A_n = \alpha^{-1} T_n^* \alpha P$ and $A = \alpha^{-1} T^* \alpha P$. Since $T_n \rightarrow T$, since $\pi_1 V$ is a closed subspace of X^* and since $\text{Range } A_n^* \subset \pi_1 V$ for each n , it follows that $A_n \rightarrow A$ and $\text{Range } A^* \subset \pi_1 V$. The mapping β is therefore onto $(\mathfrak{B}(V))^*$.

It is clear furthermore that β is one to one, linear, bicontinuous and preserves multiplication, i.e., $\beta(AB) = (\beta A)(\beta B)$ for all $A, B \in \mathfrak{A}_1$. We can thus identify $\mathfrak{A}_1$ and $(\mathfrak{B}(V))^*$.

Now, $\mathfrak{B}(V)$ is a left annihilator algebra [4, p. 107] so $(\mathfrak{B}(V))^*$ is a right annihilator algebra. It follows that $\mathfrak{A}_1$ is a right annihilator algebra.

Next, if $A \in \mathfrak{A}_1$ and $B \in \mathfrak{B}_2$ then for any $x^* \in X^*$ and $x \in X$ we have $((AB)^* x^*)x = (A^* x^*)(Bx) = 0$, since $A^* x^* \in \pi_1 V$ and $Bx \in V^+$. Therefore $(AB)^* = 0$ so that $AB = 0$, hence $\mathfrak{B}_2$ annihilates $\mathfrak{A}_1$ on the right.

We consider next $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ which is the subalgebra consisting of those operators in $\mathfrak{B}(X)$ with range in $\pi_2 X_2$ and whose adjoints have range in $\pi_1 X_1$. In order to show that $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ is a left annihilator algebra it suffices to notice that $\mathfrak{A}_1 \oplus \mathfrak{B}_1 = (\mathfrak{B}(X_2))^{**}$. This equality follows from the fact that if $T \in \mathfrak{B}(X_2)$ and $T = x_2 \otimes x_1$ with $x_2 \in X_2$ and $x_1 \in X_1 = X_2^*$

² If $\mathfrak{B}$ is a set of operators we denote by $\mathfrak{B}^*$ the set $\{T^* | T \in \mathfrak{B}\}$.

then $T^{**} = \pi_2 x_2 \otimes \pi_1 x_1 \in \mathfrak{A}_1 \oplus \mathfrak{B}_1$, so that $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ and $(\mathfrak{B}(X_2))^{**}$ contain the same one-dimensional operators.

That $\mathfrak{B}_3$ annihilates $\mathfrak{A}_1 \oplus \mathfrak{B}_1$ on the left follows from the fact that if $A \in \mathfrak{A}_1 \oplus \mathfrak{B}_1$, $B \in \mathfrak{B}_3$, $x^* \in X^*$ and $x \in X$ we have $((BA)^* x^*)(x) = (B^* x^*)(Ax) = 0$, since $B^* x^* \in (\pi_2 X_2)^+$ and $Ax \in \pi_2 X_2$, so that $(BA)^* = 0$ and therefore $BA = 0$. The same conclusion holds if $A \in \mathfrak{B}_3$, $\mathfrak{B}_3$ is therefore nilpotent.

Finally we notice that if $\mathfrak{A}_1$ is a left annihilator algebra it is an annihilator algebra and so are $(\mathfrak{B}(V))^*$ and $\mathfrak{B}(V)$, consequently by [1] V is a reflexive Banach space and so is V^* . But V^* is isomorphic with X_2 by proof of Theorem 3.1 [2]; so X_2 is reflexive and by Lemma 3 we conclude that X is reflexive. If X is reflexive then $U = V^+ = (\pi_2 X_2)^+ = (0)$ which implies $\mathfrak{B}_i = (0)$ for $i = 1, 2$ and 3 .

If $\mathfrak{B}_i = (0)$ this implies either $U = (0)$ or $V^+ = (0)$ or $(\pi_2 X_2)^+ = (0)$. But by Lemma 3 either one of these inequalities implies the other two so that $\mathfrak{B}_1 = (0)$ and $\mathfrak{A}_1 \oplus \mathfrak{B}_1 = \mathfrak{A}_1$, and hence $\mathfrak{A}_1$ is a left annihilator algebra.

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