EXISTENCE OF INVARIANT MEASURES FOR MARKOV PROCESSES[1]

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Introduction. Let X be a locally compact Hausdorff space with a countable base for its neighbourhood system. By [**5**, p. 147] the space X is a Tychonoff space and is metrizable [**5**, p. 125].

Let $P(x, A)$ be a transition function, namely: For a fixed $x \in X$, $P(x, A)$ is a measure, on the Borel subsets of X, with total mass 1. For a fixed open set A, $P(x, A)$ is continuous.

Define the operators T and S by:

$$(Tf)(x) = \int_X f(y)P(x, dy), \qquad f \in C(X),$$

$$(S\mu)(A) = \int_X P(x, A)\mu(dx),$$

where μ is a countable additive measure on the Borel subsets of X, $\mu(X) < \infty$.

If $f \in C(X)$ then Tf is continuous too. Also

$$\int_X (Tf)(x)\mu(dx) = \int_X f(x)(S\mu)(dx);$$

compare with [**4**].

The space X being a Tychonoff space has a Stone-Čech compactification, which will be denoted by X^*. See [**5**, p. 153] or [**2**, pp. 276–277].

The operator T is defined, in a natural way on $C(X^*)$ and $\|T\| = 1$, $T \geqq 0$. Also if μ is a measure on X, and therefore on X^*, then $T^*\mu = S\mu$.

Invariant measure. A measure μ, defined on X is called invariant if $S\mu = \mu$. By measure we mean positive measure; otherwise it will be called signed measure.

LEMMA 1. *Let μ be a measure on X^*. If $T^*\mu = \mu$ then the restriction of μ to X is an invariant measure.*

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PROOF. Let $\mu=\mu_1+\mu_2$ where μ_1 is the restriction of μ to X, $\mu_1(A)=\mu(A\cap X)$ and $\mu_2(A)=\mu(A\cap X^*-X)$. By assumption

$$\mu_1+\mu_2 = T^*\mu_1+T^*\mu_2 = S\mu_1+T^*\mu_2.$$

If ν is the restriction of $T^*\mu_2$ to X and $\sigma=T^*\mu_2-\nu$, then

$$\mu_1 = S\mu_1+\nu, \qquad \mu_2=\sigma.$$

Now

$$\mu_1(X) = (S\mu_1)(X)+\nu(X)$$

but

$$(S\mu_1)(X) = \mu_1(X) \quad \text{or} \quad \nu(X) = 0.$$

Thus $\nu=0$ since it is a positive measure.

LEMMA 2. *Let C be a compact subset of X. Let*

$$K = \{\mu \mid \mu \geqq 0 \text{ and } \mu(C)\geqq\delta\}, \qquad \delta>0.$$

*The set K is convex and weak * closed.*

PROOF. Let ν be in the closure of K. For every continuous positive function f which is 1 on C

$$\int f(x)\nu(dx) \geqq \delta$$

since this holds on K.

Now if U is an open set containing C there is a continuous function f with $0\leqq f\leqq 1$ and $f(x)=1$, $x\in C$, $f(x)=0$, $x\notin U$. (See [**3**, Chapter X, Theorem B].) Thus $\nu(U)\geqq\delta$ and by [**3**, Chapter X, Theorem E] $\nu(C)\geqq\delta$.

DEFINITION 1. *A set A is called dissipative if*

$$\liminf\,(S^n\mu)(A) = 0$$

for every measure μ on X. Otherwise it will be called nondissipative.

THEOREM 3. *If X contains a compact nondissipative set C, then there exists an invariant measure.*

PROOF. By assumption there is a measure μ and a positive number δ such that:

$$(S^n\mu)(C) \geqq \delta, \qquad n=0,1,2,\cdots.$$

If L is the closed convex hull of $\{S^n\mu\}$ then $\nu(C)\geqq\delta$ for every $\nu\in L$ by Lemma 2. Now the set L contains an invariant measure, σ, by Theorem V.10.5 of [2]. The restriction of this measure to X is invariant by Lemma 1 and not zero for $\sigma(C) \geqq \delta$.

In the rest of this paper we will denote $S^n\mu$ by μ^n.

Let μ be an invariant measure on X. Let $k = k(\mu)$ be its kernel (the complement of the greatest open set on which μ vanishes). Then

$$\mu(X - k) = 0 = \int_X P(x, X - k)\mu(dx) = \int_k P(x, X - k)\mu(dx).$$

Thus

$$P(x, X - k) = 0 \text{ a.e. } \quad \text{if } x \in K.$$

By continuity $P(x, X-k) = 0$ if $x \in k$ or $p,(x\ k) = 1$ for all $x \in k$.

Definition 2. *A set $A \subset X$ is called self-contained if it is closed and $P(x, A) = 1$ for all $x \in A$.*

Let $P^n(x, A)$ be the nth iterate of $P(x, A)$. Given a self-contained closed set define

$$A^n = \{x \mid P^n(x, A) > 0\}, \qquad A^* = \bigcup_{n=1}^{\infty} A^n - A.$$

On $AP^n(x, A) = 1$ for every n.

In the terminology of Markov chains, A^* consists of inessential states; see [1, p. 11].

Theorem 4. *If μ is an invariant measure and A a self-contained set, then $\mu(A^*) = 0$.*

Proof. For every $n \geqq 1$

$$\begin{aligned}\mu(A) &= \mu^n(A) = \int_X P^n(x, A)\mu(dx) \\ &= \int_A P^n(x, A)\mu(dx) + \int_{A^n - A} P^n(x, A)\mu(dx) \\ &= \mu(A) + \int_{A^n - A} P^n(x, A)\mu(dx).\end{aligned}$$

Thus $\mu(A^n - A) = 0$.

Lemma 5. *If A is self-contained so is $B = X - A^*$.*

Proof. The set B is closed and $P(x, X-A) = 1$ if $x \in B$. It is enough to show that $P(x, A^n) = 0$ for $x \in B$. Now

$$0 = P^{n+1}(x, A) = \int_X P(x, dy)P^n(y, A) = \int_{A^n} P(x, dy)P^n(y, A).$$

Thus

$$P(x, A^n) = 0 \quad \text{for} \quad P^n(y, A) > 0, \qquad y \in A^n.$$

Let us consider the set of all collections $\{\sigma_\alpha\}$ (of invariant probability measures) with the property that $k(\alpha_1) \cap k(\alpha_2) = 0$ if $\alpha_1 \neq \alpha_2$. Order this set by inclusion. By Zorn's Lemma there is a maximal element, which we will denote by $\{\mu_\alpha\}$.

LEMMA 6. *The set $\{\mu_\alpha\}$ is countable.*

PROOF. One can extract a countable set $\mu_i = \mu_{\alpha_i}$ such that

$$\bigcup (k^*(\mu_i) \cup k(\mu_i)) = \bigcup (k^*(\mu_\alpha) \cup k(\mu_\alpha)).$$

This is possible because the space X is separable. For every μ_α

$$\mu_\alpha(X - \bigcup (k^*(\mu_i) \cup k(\mu_i))) = 0$$

by choice of μ_i. Also

$$\mu_\alpha(k^*(\mu_i)) = 0$$

by Theorem 4. Thus for some i, $\mu_\alpha(k(\mu_i)) \neq 0$ and therefore $\mu_\alpha = \mu_i$.

Let $\mu = \sum \epsilon_i \mu_i$ where $\epsilon_i > 0$, $\sum \epsilon_i = 1$. Then $k(\mu) = (\bigcup k(\mu_i))^-$. Denote $X_1 = k(\mu)$, $X_2 = k^*(\mu)$, $X_3 = X - X_1 \cup X_2$.

The sets X_1 and X_3 are self-contained and on X_2 every invariant measure vanishes.

THEOREM 6. *If σ is an invariant measure then $\sigma(X_3) = 0$.*

PROOF. Because $\sigma(X_2) = 0$ and X_1, X_3 are self-contained, the restriction of σ to X_3 is invariant too. Now if $\sigma(X_3) \neq 0$, then σ restricted to X_3 would extend the collection $\{\mu_i\}$, which was assumed maximal.

The set X_1 is thus uniquely defined as the union of all kernels of invariant measures. Therefore X_2 and X_3 are uniquely defined too.

THEOREM 7. *Every compact subset of X_3 is dissipative.*

PROOF. This follows immediately from Theorem 3.

REMARK. It is not known to us whether or not $\lim \mu^n(C) = 0$ for every compact subset of X_3. For Markov chains this is known.

In order to get uniqueness of the invariant measure, it seems reasonable to assume that X contains no proper subsets which are self-contained. First we will need a result on signed measures. If σ is a signed measure then $\sigma = \sigma_+ - \sigma_-$ where σ_+ is a measure defined on A, σ_- a measure on B where $A \cup B = X$ and $A \cap B = \emptyset$. See [**3**, p. 123].

LEMMA 8. *Let $\sigma = S\sigma$, then both σ_+ and σ_- are invariant measures.*

PROOF. By definition $\sigma_+(B)=\sigma_-(A)=0$. Hence

$$\begin{aligned}\sigma(A) &= \sigma_+(A) = \int_X P(x, A)\sigma(dx)\\ &= \int_A P(x, A)\sigma_+(dx) - \int_B P(x, B)\sigma_-(dx)\\ &\leqq \sigma_+(A) - \int_B P(x, A)\sigma_-(dx).\end{aligned}$$

Thus

$$\int_B P(x, A)\sigma_-(dx) = 0.$$

Now if $C\subset A$ then

$$\sigma(C) = \sigma_+(C) = \int_A P(x, C)\sigma_+(dx) - \int_B P(x, C)\sigma_-(dx)$$

but

$$\int_B P(x, C)\sigma_-(dx) \leqq \int_B P(x, A)\sigma_-(dx) = 0.$$

Also if $C\subset B$ then $\sigma_+(C)=0$ and

$$\int_X P(x, C)\sigma_+(dx) \leqq \int_A P(x, B)\sigma_+(dx)$$

and this is zero by the argument used above applied to $-\sigma$.

THEOREM 9. *If X does not contain any proper self-contained subsets, then there is at most one invariant measure.*

PROOF. Let μ_1 and μ_2 be invariant measures. Define $\mu_1-\mu_2=\sigma=\sigma_+-\sigma_-$. Let A and B be as in the previous Lemma. Then $k(\sigma_+)=k(\sigma_-)=X$ by assumption. Now

$$\sigma_+(A) = \int_A P(x, A)\sigma_+(dx)$$

or $P(x, A)=1$ a.e. on A with respect to σ_+. Similarly $P(x, A)=0$ a.e. on B with respect to σ_-. Let $x\in A$ be such that $P(x, A)=1$; then there is a closed set $A_1\subset A$ such that $P(x, A_1)>1/2$. See [2, III.9.22]. Now every neighbourhood of x has a positive σ_- measure for $k(\sigma_-)=X$.

Thus every neighbourhood of x contains points y such that $y \in B$ and $P(y, A) = 0$. Hence

$$0 \leqq P(y, A_1) \leqq P(y, A) = 0.$$

But this contradicts the continuity of $P(x, A)$.

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DARBOUX FUNCTIONS OF BAIRE CLASS ONE AND DERIVATIVES

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Introduction. Let $I_0 = [0, 1]$ and let R be the reals. Let B_1 be the class of functions $f: I_0 \to R$ of Baire type at most one, and denote by D the class of functions $f: I_0 \to R$ which possess the Darboux property, i.e., take connected sets into connected sets. The class $B_1 \cap D$ is abbreviated by (B_1, D). If Δ is the class of functions $f: I_0 \to R$ which are derivatives, then we have the well-known relation $\Delta \subset (B_1, D)$. It is of interest to have characterizations for the classes Δ and (B_1, D). In this paper two characterizations of (B_1, D) are given as well as a characterization of Δ. This characterization together with one characterization of (B_1, D) provides a measurement by how much a function in (B_1, D) may fail to be in Δ.

Throughout the paper we will use the following notation. For $A \subset I_0$, A^0 is the interior of A relative to I_0, $\overline{A}$ stands for the closure of A, and $|A|$ denotes the Lebesgue measure of A.

First characterization of (B_1, D). We have occasion to use the following characterizations of B_1. (1) $f \in B_1$ if and only if for each $a \in R$ the sets $\{x : f(x) \geqq a\}$, $\{x : f(x) \leqq a\}$ are G_δ; (2) $f \in B_1$ if and only if every perfect subset P of I_0 has a point of continuity of $f|P$ (f restricted to P) [**3**].

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