ON THE NUMERICAL RANGE OF AN OPERATOR

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The numerical range of an operator T in a Hilbert space is defined as the set of all the complex numbers (Tx, x) , where x is a unit vector in the space. It is well known that a bounded normal operator has the property that the closure of its numerical range is exactly the convex hull of its spectrum [5, pp. 325–327, Theorem 8.13 and Theorem 8.14]. Call this property A. In this article let T denote a linear bounded operator in a Hilbert space H , $V(T)$ be its numerical range, $K(T)$ be the convex hull of its spectrum, and use the usual notations $\rho(T)$, $\sigma(T)$, $P\sigma(T)$, $C\sigma(T)$, $R\sigma(T)$ respectively for the resolvent set, spectrum, point spectrum, continuous spectrum, and residual spectrum of T . This article shall investigate some consequences of the property A.

In general T having the property A need not be normal. Hence little can be said about an operator unless something else is known. Define a property B if $\sigma(T)$ lies on a convex curve. Note that a class of operators such as unitary operators have this property. The following theorems shall be proved:

THEOREM 1. *In a finite dimensional vector space T having the property B is normal if and only if it has the property A.*

THEOREM 2. *If T has the properties A and B, it can be expressed as a direct sum $T_1 \oplus T_2$ defined on a product space $H_1 \oplus H_2$, where H_1 is spanned by all the characteristic elements of T , such that:*

- (a) T_1 is normal, and $\sigma(T_1) = \overline{P\sigma(T_1)}$,
- (b) $\sigma(T_2) = C\sigma(T_2)$,
- (c) T is normal if and only if T_2 is normal.

THEOREM 3. *If a compact operator T has the properties A and B, then with the same notations used in Theorem 2:*

- (a) T_2 is quasi-nilpotent,
- (b) T is a spectral operator of class Γ in the sense of Dunford [2, p. 325],
- (c) T is normal if and only if $H = H_1$.

We shall need the following lemmas for the proofs of the theorems:

LEMMA 1. *If p, q are any fixed complex numbers, b any fixed nonzero number, then the interior of the ellipse with its foci at p, q and with a minor axis $|b|$ is contained in the set of numbers of the form:*

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$$s = \alpha \bar{\alpha} p + \beta \bar{\beta} q + \alpha \beta b$$

for all α, β satisfying $\alpha \bar{\alpha} + \beta \bar{\beta} = 1$.

PROOF. For a fixed $0 \leq c \leq 1$, and $|\alpha| = c$, s describes a circle with its center on the line segment connecting p, q and with a radius $c(1-c^2)^{1/2}|b|$. Hence for all such c , s describes a family of circles the envelope of which is the ellipse mentioned in the lemma. This completes the proof. It should be remarked that p and q are interior points of the ellipse.

LEMMA 2. Let a_1, a_2 be distinct points in $P\sigma(T)$, and L_1, L_2 be the corresponding characteristic manifolds. If L_1 and L_2 are not orthogonal to each other, then T is not normal, and a_1 and a_2 are interior points of $V(T)$.

PROOF. T is not normal because for a normal operator all the characteristic manifolds are mutually orthogonal. Let $\{s_i\}, \{t_i\}$ be orthonormal bases for L_1 and L_2 respectively. By hypothesis there exist one s_i and one t_i , say s_1 and t_1 , such that $(s_1, t_1) = de^{i\Delta} = \delta$, where $0 < d < 1$. The second inequality is true, because $|(s_1, t_1)| \leq \|s_1\| \|t_1\| = 1$, and if the equality held true in the Schwarz inequality, then s_1 and t_1 would be proportional, which is impossible.

Now consider the subspace L spanned by s_1 and t_1 . Clearly the numerical range of T on L is contained in $V(T)$. Take a unit vector u in L which may be expressed as:

$$u = \left[\bar{\alpha} - \frac{\beta \bar{\delta}}{(1-d^2)^{1/2}} \right] s_1 + \frac{\beta}{(1-d^2)^{1/2}} t_1$$

where $\alpha \bar{\alpha} + \beta \bar{\beta} = 1$. Then:

$$(Tu, u) = \alpha \bar{\alpha} a_1 + \beta \bar{\beta} a_2 + \frac{\alpha \beta \bar{\delta} (a_2 - a_1)}{(1-d^2)^{1/2}},$$

where $\bar{\delta}(a_2 - a_1)/(1-d^2)^{1/2}$ is nonzero. Therefore by Lemma 1, a_1 and a_2 are interior points of $V(T)$.

LEMMA 3. If T has the properties A and B, all its characteristic manifolds are mutually orthogonal.

PROOF. This is evident from Lemma 2 because each point in $P\sigma(T)$ must lie on the boundary of $K(T)$ which is $\overline{V(T)}$, and hence cannot be an interior point of $V(T)$.

LEMMA 4. $R\sigma(T)$ lies in the interior of $V(T)$.

PROOF. Take any point a in $R\sigma(T)$. Then the closure of $(T - aI)H$

is a proper subspace of H . Take any unit vector x in its orthogonal complement; then $((T-aI)x, x) = 0 = (Tx, x) - a$. But $(T-aI)x = bx_1$ for some unit vector x_1 with a nonzero b . Consider the set of all the unit vectors u of the form: $u = \alpha x + \beta x_1$ where $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$. Then $(Tu, u) = \alpha\bar{\alpha}a + \beta\bar{\beta}(Tx_1, x_1) + \alpha\bar{\beta}b$. Hence by Lemma 1 a is in the interior of $V(T)$.

As a consequence of Lemma 4, we conclude:

LEMMA 5. $R\sigma(T)$ is empty if T has the properties A and B.

Denote by H_1 the smallest subspace of H which contains all the characteristic manifolds of T , and by H_2 the orthogonal complement of H_1 . Denote by L_a the characteristic manifold corresponding to the point a in $P\sigma(T)$. Then by Lemma 3, and by definition, [4, p. 8], H_1 is the direct sum of $\{L_a\}$, $a \in P\sigma(T)$, i.e., $H_1 = \sum \oplus L_a$ over all a in $P\sigma(T)$, when T has the properties A and B, and any element x in H_1 can be expressed as: $x = \sum_{i=1}^n c_i x_i$, where x_i is a unit vector in L_{a_i} , and $\{a_i\}$ are all distinct.

LEMMA 6. T is reducible by H_1 if it has the properties A and B.

PROOF. It is evident that $T(H_1) \subseteq H_1$. Hence we only need to show $T(H_2) \subseteq H_2$, [4, p. 33]. Suppose the contrary; then there exist a unit vector x in H_1 corresponding to some characteristic value a , and a unit vector y in H_2 such that $(Ty, x) \neq 0$. Consider the set of unit vectors u of the form $u = \bar{\alpha}x + \beta y$, where $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$. Then $(Tu, u) = \alpha\bar{\alpha}a + \beta\bar{\beta}(Ty, y) + \alpha\bar{\beta}(Ty, x)$. By Lemma 1, a is an interior point of $V(T)$, which contradicts the properties A and B.

Let T_1, T_2 be the parts of T restricted on H_1 and H_2 respectively. From the lemma we see that T sends H_1 into itself, if T has the properties A and B. Hence T_1 may be considered as an endomorphism on H_1 . Similar conclusion can be made for T_2 .

PROOF OF THEOREM 1. Necessity is evident. The sufficiency can be proved as follows. Since the space is of finite dimension, $\sigma(T) = P\sigma(T)$ and $H = H_1$. By Lemma 3 the characteristic manifolds corresponding to distinct characteristic values are mutually orthogonal. Hence T is normal because $T = \sum_{i=1}^m a_i P_i$, where $\{a_i\}$ are the characteristic values and P_i the projection operator of the corresponding manifold of a_i .

In order to take care of the cases where H_1 or H_2 is empty, define the spectrum of an operator on an empty subspace to be the empty set.

PROOF OF THEOREM 2. Clearly $P\sigma(T) = P\sigma(T_1)$. By Lemma 3, T_1 is normal, [4, p. 33]. Also, $R\sigma(T_1)$ is empty, [5, p. 325]. Consider

any complex number s which is not in the closure of $P\sigma(T_1)$. Let $d > 0$ be such that $|s - a| \geq d$ for all a in $\overline{P\sigma(T_1)}$ and I_i denote the identity operator on H_i , $i = 1, 2$. It is easy to see that the image of the bounded set of elements in H_1 with the bound $1/d$ under the mapping $(T_1 - sI_1)$ contains the unit sphere of H_1 , and hence that the bounded inverse of $(T_1 - sI_1)$ exists for every such s . Therefore $\sigma(T_1) = \overline{P\sigma(T_1)}$.

By means of Lemma 4 with T_2, H_2 in place of T, H , it is evident that $R\sigma(T_2)$ is empty because $V(T_2) \subseteq V(T)$. Then part (b) is obvious if I can show that $P\sigma(T_2)$ is empty. Suppose that there exist s in $P\sigma(T_2)$. Then a nonzero element y in H_2 exists with $T_2 y = sy$. Now $Ty = (T_1 \oplus T_2)y = T_1 \theta + T_2 y = sy$. Thus $s \in P\sigma(T)$ and $y \in H_1$, which contradicts the fact that H_1 and H_2 are orthogonal to each other. Hence $P\sigma(T_2)$ is empty, and consequently $\sigma(T_2) = C\sigma(T_2)$.

Now $T = T_1 \oplus T_2$, hence T is normal if and only if T_2 is normal. This completes the proof. It should be remarked that as a consequence of this theorem, $R\sigma(T)$ is empty because $R\sigma(T) \subseteq R\sigma(T_1) \cup R\sigma(T_2)$.

The following corollary is an immediate consequence of Theorem 2:

COROLLARY 1. *With the same hypothesis as that in Theorem 2, T is normal if H is spanned by the characteristic elements of T .*

Denote by θ_i the zero operators on H_i , $i = 1, 2$, and define $S_1 = T_1 \oplus \theta_2$, $S_2 = \theta_1 \oplus T_2$. Then $T = S_1 + S_2$. Since the decomposition of H into H_1 and H_2 is unique for T having the properties A and B, we conclude:

COROLLARY 2. *For a linear bounded operator T on a Hilbert space H having the properties A and B there exists a unique decomposition of T as a sum of two linear bounded operators S_1 and S_2 such that:*

(a) $T = S_1$ on the subspace spanned by all the characteristic elements of T ,

(b) $T = S_2$ on the orthogonal complement of the above subspace,

(c) T is normal if and only if S_2 is normal.

A linear bounded operator is said to be a compact operator if it maps the unit sphere into a conditionally compact set. Since a bounded closed subset of a compact set is compact, we have the following lemma which shall be needed in the proof of Theorem 3:

LEMMA 7. *If T is compact, and if a subspace reduces T , the reduced part of T on the subspace is compact.*

PROOF OF THEOREM 3. Part (a) is obvious if H_2 is empty. Otherwise by the above lemma T_2 is compact. Theorem 2 implies $\sigma(T_2) = C\sigma(T_2)$, and hence $\sigma(T_2) = \{0\}$, [1, p. 160, Theorem 22]. Thus the inverse of

$T_2 - aI_2$ exists for all nonzero a , and $(T_2 - aI_2)^{-1}$ is an entire function of a^{-1} . Consequently T_2 is quasi-nilpotent, [3, p. 98 and p. 107]. It is not nilpotent because it would contradict the fact that $P\sigma(T_2)$ is empty.

By Corollary 2 to Theorem 2 T can be uniquely decomposed as the sum of S_1 and S_2 where S_1 is normal and hence is a scalar operator [2, p. 332]. Now $S_2 = \theta_1 \oplus T_2$ implies $S_2^n = \theta_1 \oplus T_2^n$. Then $\lim_{n \rightarrow \infty} \|S_2^n\|^{1/n} = \lim_{n \rightarrow \infty} \|T_2^n\|^{1/n} = 0$. Hence S_2 is also quasi-nilpotent. Clearly, S_1, S_2 commute with each other. Then T is a spectral operator of class Γ , [2, p. 333, Theorem 8].

By Theorem 2 we see that T is normal if and only if T_2 is normal. However, if T_2 is normal then $\|T_2^n\|^{1/n} = \|T_2\|$ for all n . Now T_2 is quasi-nilpotent, and then $\|T_2\| = \lim_{n \rightarrow \infty} \|T_2^n\|^{1/n} = 0$ which implies $T_2 = \theta_2$. But $P\sigma(T_2)$ is empty; hence this causes a contradiction unless H_2 is empty, i.e., $H = H_1$. This completes the proof of Theorem 3.

The following result is an immediate consequence of the theorem:

COROLLARY 3. *A compact operator on a Hilbert space, having the property B, is normal if and only if it satisfies the property A and the space is spanned by all its characteristic elements.*

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