THE RELATION BETWEEN CONTINUITY AND DIFFERENTIABILITY OF FUNCTIONS ON ALGEBRAS

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1. Introduction. Let $\mathfrak{A}$ be a finite dimensional associative algebra with an identity over the real or complex field $\mathfrak{F}$, and let f be a function on $\mathfrak{A}$ to $\mathfrak{A}$, i.e., a function with domain and range in $\mathfrak{A}$. $\mathfrak{A}$ is a normable ring, hence a metric topological space in the metric induced by the chosen norm [4]. Consequently, the usual elementary concepts of limits and continuity make sense, and the customary elementary theorems are valid.

In [2] a generalized difference quotient definition of differentiability and derivative was given for a function f for the case of a total matrix algebra over $\mathfrak{F}$. This definition, which is equally applicable to any finite dimensional associative algebra $\mathfrak{A}$ over $\mathfrak{F}$, is:

DEFINITION. Let $f(\xi)$ be a function with domain and range in $\mathfrak{A}$ defined in some neighborhood of $\xi = \alpha$. Then $f(\xi)$ is said to be differentiable at $\xi = \alpha$, if, for all $\delta \in \mathfrak{A}$ in a sufficiently small neighborhood $\mathfrak{N}$ of 0,

(I) the difference $f(\alpha + \delta) - f(\alpha)$ is expressible as a finite sum of the form $f(\alpha + \delta) - f(\alpha) = \sum_{i=1}^k \lambda_i \delta \mu_i$ where $\lambda_i, \mu_i \in \mathfrak{A}$, and

(II) $\lim_{\delta \rightarrow 0} \sum_{i=1}^k \lambda_i \mu_i$ exists.

If I and II are fulfilled, then the limit in II is called the derivative of $f(\xi)$ at $\xi = \alpha$, and is denoted by $f^I(\alpha)$.

If $\mathfrak{A}$ is commutative, it is easily verified that the above definition implies that $f(\xi)$ is Fréchet-differentiable at $\xi = \alpha$, which in turn implies that $f(\xi)$ is continuous at $\xi = \alpha$ [1]. For noncommutative $\mathfrak{A}$ this inference is not warranted.

The proofs in [2] of the uniqueness of the derivative, and of the theorems concerning differentiability and derivative of the sum or product of two functions, are equally valid for the more general algebras $\mathfrak{A}$ of the type considered here. However, the proof that the product fg of two functions is differentiable at $\xi = \alpha$ if f and g are, and that $(fg)^I(\alpha) = f^I(\alpha)g(\alpha) + f(\alpha)g^I(\alpha)$, assumed that at least one of the functions f, g was continuous at $\xi = \alpha$, and it was conjectured that (A) *this hypothesis is essential* since (B) *differentiability at a point does not imply continuity at the point*.

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In this note the above double conjecture is settled for any $\mathfrak{A}$ as defined above by disproving Conjecture A, and proving Conjecture B, together with some theorems which get at the essential reason for the validity of B and settle a question of N. J. Fine.

2. Differentiability of the product of differentiable functions. As a preliminary to the treatment of Conjecture A, the definition of differentiability is recast as follows.

LEMMA 2.1. *A function $f(\xi)$ defined in a neighborhood of $\xi = \alpha \in \mathfrak{A}$, is differentiable at $\xi = \alpha$ if and only if, for all $\delta \in \mathfrak{A}$ with sufficiently small norm, the functional difference can be written in the form*

$$f(\alpha + \delta) - f(\alpha) = \delta\Phi + \sum_{i=1}^m \nu_i \delta \eta_i, \quad (\Phi, \nu_i, \eta_i \in \mathfrak{A})$$

where $\lim_{\delta \rightarrow 0} \Phi$ exists and

$$\sum_{i=1}^m \nu_i \eta_i = 0.$$

PROOF. The sufficiency of the condition follows directly from the definition. For the necessity, we observe that from condition I of the definition, for $\|\delta\|$ sufficiently small, $f(\alpha + \delta) - f(\alpha) = \delta \sum_{i=1}^k \lambda_i \mu_i + [\sum_{i=1}^k \lambda_i \delta \mu_i - \delta \sum_{i=1}^k \lambda_i \mu_i]$. Since $\mathfrak{A}$ has an identity, the bracketed expression is of the form $\sum_{i=1}^m \nu_i \delta \eta_i$. Further $\sum_{i=1}^m \nu_i \eta_i = 0$, and condition II of the definition assures that $\lim_{\delta \rightarrow 0} \sum_{i=1}^k \lambda_i \mu_i$ exists.

Conjecture A is now settled in the negative by

THEOREM 2.1. *Let $f(\xi)$ and $g(\xi)$ be functions differentiable at $\xi = \alpha \in \mathfrak{A}$. Then $p(\xi) = f(\xi)g(\xi)$ is differentiable at $\xi = \alpha$ and $p^I(\alpha) = f^I(\alpha)g(\alpha) + f(\alpha)g^I(\alpha)$.*

PROOF. By the lemma

$$f(\alpha + \delta) = f(\alpha) + \delta\Phi + \sum_{i=1}^k \nu_i \delta \eta_i,$$

$$g(\alpha + \delta) = g(\alpha) + \delta\Psi + \sum_{i=1}^m \theta_i \delta \pi_i$$

such that $\lim_{\delta \rightarrow 0} \Phi = f^I(\alpha)$, $\lim_{\delta \rightarrow 0} \Psi = g^I(\alpha)$, and

$$(2.1) \quad \sum_{i=1}^k \nu_i \eta_i = \sum_{i=1}^m \theta_i \pi_i = 0 \text{ for all } \delta \in \mathfrak{A} \text{ in some neighborhood of } 0.$$

Therefore,

$$\begin{aligned}
 p(\alpha + \delta) - p(\alpha) &= f(\alpha + \delta)g(\alpha + \delta) - f(\alpha)g(\alpha) \\
 &= \left[f(\alpha) + \delta\Phi + \sum_{i=1}^k \nu_i \delta\eta_i \right] \left[g(\alpha) + \delta\Psi + \sum_{i=1}^m \theta_i \delta\pi_i \right] \\
 &\quad - f(\alpha)g(\alpha) \\
 &= f(\alpha)\delta\Psi + f(\alpha) \sum_{i=1}^m \theta_i \delta\pi_i + \delta\Phi g(\alpha) + \delta\Phi\delta\Psi \\
 &\quad + \delta\Phi \sum_{i=1}^m \theta_i \delta\pi_i + \left[\sum_{i=1}^k \nu_i \delta\eta_i \right] \delta\Psi \\
 &\quad + \left[\sum_{i=1}^k \nu_i \delta\eta_i \right] g(\alpha) + \left[\sum_{i=1}^k \nu_i \delta\eta_i \right] \left[\sum_{i=1}^m \theta_i \delta\pi_i \right].
 \end{aligned}$$

Thus the difference $p(\alpha + \delta) - p(\alpha)$ satisfies condition I of the definition of differentiability. To verify condition II, we need to establish the existence of the limit of a "detached coefficient" of δ . Any such coefficient will do, by virtue of the uniqueness of the derivative, when it exists [2]. One such "detached coefficient" is,

$$\begin{aligned}
 f(\alpha)\Psi + \Phi g(\alpha) + \delta\Phi\Psi + f(\alpha) \sum_{i=1}^m \theta_i \pi_i + \delta\Phi \sum_{i=1}^m \theta_i \pi_i + \left[\sum_{i=1}^k \nu_i \eta_i \right] \delta\Psi \\
 + \left[\sum_{i=1}^k \nu_i \eta_i \right] g(\alpha) + \left[\sum_{i=1}^k \nu_i \eta_i \right] \left[\sum_{i=1}^m \theta_i \delta\pi_i \right]
 \end{aligned}$$

which by equations 2.1, is

$$f(\alpha)\Psi + \Phi g(\alpha) + \delta\Phi\Psi.$$

The limit as $\delta \rightarrow 0$ of this expression exists, since the limits of Φ and Ψ exist, and that limit is $f(\alpha)g^I(\alpha) + f^I(\alpha)g(\alpha)$.

3. Differentiability does not imply continuity. Theorem 2.1 was proved without invoking continuity of f or g at α as in [2]. Indeed, the following example, communicated by N. J. Fine, shows that a function may be differentiable at $\alpha \in \mathfrak{A}$ and discontinuous at α .

Let $\mathfrak{A}$ be the algebra of 2×2 matrices over the real field. For

$$\xi = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

let $M_\xi = \max_{i,j} |x_{ij}|$. Define a function $f(\xi)$ by,

$$f(\xi) = \begin{pmatrix} x_{11} & M_\xi^2 x_{12} \\ x_{21}/M_\xi^2 & x_{22} \end{pmatrix}, \quad \text{for } \xi \neq 0$$

$$f(0) = 0.$$

$f(\xi)$ is discontinuous at $\xi=0$, since $\lim_{\xi \rightarrow 0} f(\xi)$ does not exist. However,

$$f(0 + \delta) - f(0) = f(\delta) = P^{-1}\delta P$$

where

$$P = \begin{pmatrix} M_\delta & 0 \\ 0 & 1/M_\delta \end{pmatrix} \quad \text{for } \delta \neq 0,$$

$$P = I, \text{ the identity matrix,} \quad \text{for } \delta = 0.$$

Hence f fulfills condition I for differentiability at $\xi=0$. Further $\lim_{\delta \rightarrow 0} PP^{-1} = I$. Hence $f(\xi)$ is differentiable at $\xi=0$.

The example is, so to speak, a pointwise example, and Fine, in the cited communication, raised the question: Is a function which is differentiable in a neighborhood of $\alpha \in \mathfrak{A}$, necessarily continuous at α ? That this is not the case, and why it is not, is shown by the following theorem.

THEOREM 3.1. *Let $\epsilon_1, \dots, \epsilon_n$ be a basis for $\mathfrak{A}$, with ϵ_1 the identity of $\mathfrak{A}$. With $\xi = \sum_{i=1}^n x_i \epsilon_i$, $x_i \in \mathfrak{F}$, let $f(\xi) = \sum_{i=1}^n f_i(x_1, \dots, x_n) \epsilon_i$, $f_i(x_1, \dots, x_n) \in \mathfrak{F}$, be defined in a neighborhood $\mathfrak{N}$ of $\alpha = \sum_{i=1}^n a_i \epsilon_i$, $a_i \in \mathfrak{F}$. A necessary condition that $f(\xi)$ be differentiable at $\xi = \alpha$ is that $\partial f_i / \partial x_1$ exist at $\xi = \alpha$ for $i = 1, \dots, n$. In this case*

$$f^I(\alpha) = \sum_{i=1}^n \left[\frac{\partial f_i}{\partial x_1} \right]_{\xi=\alpha} \epsilon_i.$$

Further, if $\mathfrak{A}$ is normal simple over $\mathfrak{F}$, then this condition is also sufficient.

PROOF. (a) *Necessity.* Since $f(\xi)$ is differentiable at $\xi = \alpha$,

$$f(\alpha + \delta) - f(\alpha) = \sum_{i=1}^k \lambda_i \delta \mu_i$$

and $\lim_{\delta \rightarrow 0} \sum_{i=1}^k \lambda_i \mu_i$ exists. Choose $\delta = d\epsilon_1$, $d \in \mathfrak{F}$. Then

$$\lim_{d \rightarrow 0} [f(\alpha + d\epsilon_1) - f(\alpha)]/d = \lim_{d \rightarrow 0} \sum_{i=1}^k \lambda_i \mu_i = f^I(\alpha)$$

exists. This implies that the limit of each basis component exists, i.e.,

$$\lim_{d \rightarrow 0} \frac{f_i(a_1 + d, a_2, \dots, a_n) - f_i(a_1, a_2, \dots, a_n)}{d} = \left. \frac{\partial f_i}{\partial x_1} \right|_{\xi=\alpha}$$

exists. Hence

$$f^I(\alpha) = \sum_{i=1}^n \left. \frac{\partial f_i}{\partial x_1} \right|_{\xi=\alpha} \epsilon_i.$$

(b) *Sufficiency, under the added hypothesis that $\mathfrak{A}$ is normal simple.*

Let $\delta = \sum_{i=1}^n d_i \epsilon_i$, $d_i \in \mathfrak{F}$, such that $(\alpha + \delta) \in \mathfrak{N}$. We seek to show first that the difference

$$f(\alpha + \delta) - f(\alpha) = \sum_{i=1}^n [f_i(\alpha + \delta) - f_i(\alpha)] \epsilon_i,$$

where $f_i(\xi)$ means $f_i(x_1, \dots, x_n)$, can be written in the form $\sum_{i=1}^k \lambda_i \delta \mu_i$, or expressing δ , λ_i , μ_i in terms of the basis elements, that

$$(3.1) \quad f(\alpha + \delta) - f(\alpha) = \sum_{i,j=1}^n t_{ij} \epsilon_i \left(\sum_{k=1}^n d_k \epsilon_k \right) \epsilon_j.$$

If c_{ijk} are the multiplication constants for the basis $\epsilon_1, \dots, \epsilon_n$, i.e., $\epsilon_i \epsilon_j = \sum_{k=1}^n c_{ijk} \epsilon_k$, then 3.1 can be written

$$(3.2) \quad f(\alpha + \delta) - f(\alpha) = \sum_{i,j,k,r=1}^n t_{ij} d_k c_{iks} c_{sjr} \epsilon_r.$$

Equation 3.2 will be fulfilled for arbitrary δ in $\mathfrak{N}$ i.e., arbitrary d_k , if and only if the system of linear equations over $\mathfrak{F}$,

$$(3.3) \quad f_r(\alpha + \delta) - f_r(\alpha) = \sum_{s,i,j=1}^n t_{ij} c_{iks} c_{sjr}, \quad k, r = 1, \dots, n$$

is solvable for the coefficients t_{ij} . Now $f(\alpha + \delta) - f(\alpha)$ can be written in the form

$$f(\alpha + \delta) - f(\alpha) = \sum_{r=1}^n [f_r(\alpha + \delta) - f_r(\alpha)] \epsilon_r = \sum_{r=1}^n \left(\sum_{k=1}^n g_{rk} d_k \right) \epsilon_r,$$

where

$$\begin{aligned} g_{rs} &= [f_r(a_1 + d_1, \dots, a_s + d_s, a_{s+1}, \dots, a_n) \\ &\quad - f_r(a_1 + d_1, \dots, a_{s-1} + d_{s-1}, a_s, \dots, a_n)] d_s^{-1}, \quad \text{if } d_s \neq 0, \\ g_{rs} &= 0, \quad \text{if } s \neq 1 \text{ and } d_s = 0, \\ g_{r1} &= \frac{\partial f_r}{\partial x_1}, \quad \text{if } d_1 = 0. \end{aligned}$$

Hence the problem of satisfying 3.2 for arbitrary d_k is transformed to that of showing that the system of linear equations over $\mathfrak{F}$,

$$(3.4) \quad \sum_{s,i,j} t_{ij} c_{iks} c_{sjr} = g_{rk} \quad (r, k = 1, \dots, n),$$

is solvable for the t_{ij} . Since $\mathfrak{A}$ is normal simple, the coefficient matrix of the t_{ij} is nonsingular [3], and 3.4 has a unique solution for the t_{ij} . Hence

$$\begin{aligned} f(\alpha + \delta) - f(\alpha) &= \sum_{r,k}^n g_{rk} d_k \epsilon_r \\ &= \sum_{i,j,k,r,s=1}^n t_{ij} c_{iks} c_{sjr} d_k \epsilon_r \\ &= \sum_{i,j=1}^n t_{ij} \epsilon_i \delta \epsilon_j \end{aligned}$$

and the function f fulfills requirement I of the definition for $\alpha + \delta$ in $\mathfrak{A}$.

Now $\lim_{\delta \rightarrow 0} \sum_{i,j=1}^n t_{ij} \epsilon_i \epsilon_j$ exists if and only if the limit of its r th component, $r = 1, \dots, n$, exists. The r th component is $\sum_{i,j=1}^n t_{ij} c_{ijr}$, $= \sum_{i,j,k=1}^n t_{ij} c_{ikl} c_{kjr}$, since $c_{ikl} = 1$ if $k = i$, 0 otherwise, and

$$\begin{aligned} \lim_{\delta \rightarrow 0} \sum_{i,j} t_{ij} c_{ikl} c_{kjr} &= \lim_{\delta \rightarrow 0} g_{r1} = \lim_{d_1 \rightarrow 0} g_{r1} \\ &= \lim_{d_1 \rightarrow 0} \frac{f_r(a_1 + d_1, a_2, \dots, a_n) - f_r(a_1, \dots, a_n)}{d_1} \\ &= \left. \frac{\partial f_r}{\partial x_1} \right|_{\xi = \alpha}. \end{aligned}$$

This holds for each $r = 1, \dots, n$, hence $f(\xi)$ is differentiable at $\xi = \alpha$.

Theorem 3.1 shows that for a normal simple algebra $\mathfrak{A}$, e.g., a total matrix algebra, differentiability of $f(\xi)$ at $\xi = \alpha$, requires only that the component functions of $f(\xi)$ be differentiable with respect to the identity component of ξ ; they may be any functions whatever, continuous or not, of $x_2, \dots, x_n$, provided only that $f(\xi)$ is defined in $\mathfrak{A}$.

Normal simple algebras are the most general algebras satisfying the second part of Theorem 3.1 in the following sense.

THEOREM 3.2. *If $\mathfrak{A}$ is not normal simple, then there exists at least one function on $\mathfrak{A}$ to $\mathfrak{A}$, with analytic component functions, which is not differentiable at any element of $\mathfrak{A}$.*

PROOF. Suppose $\mathfrak{A}$ is not simple. Then $\mathfrak{A}$ contains a nontrivial two-sided ideal $\mathfrak{C}$. The identity ϵ_1 of $\mathfrak{A}$ does not belong to $\mathfrak{C}$. Since $\mathfrak{C} \neq 0$,

$\exists$ an element $\epsilon_2 \in \mathfrak{C}$ independent of ϵ_1 over $\mathfrak{F}$. Then $\mathfrak{A}$ has a basis $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ including ϵ_1 and ϵ_2 . The function $f(\xi) = x_2 \epsilon_1$ has component functions analytic for all $\xi \in \mathfrak{A}$. If $f(\xi)$ were differentiable at some $\alpha \in \mathfrak{A}$, then for all δ with $\|\delta\|$ sufficiently small,

$$f(\alpha + \delta) - f(\alpha) = \sum_{i,j=1}^n t_{ij} \epsilon_i \delta \epsilon_j.$$

Choose $\delta = d_2 \epsilon_2 \in \mathfrak{C}$ with $d_2 \neq 0$ in $\mathfrak{F}$, and $|d_2|$ appropriately small. Then for all such d_2 ,

$$f(\alpha + d_2 \epsilon_2) - f(\alpha) = d_2 \epsilon_1 = \sum_{i,j=1}^n t_{ij} \epsilon_i d_2 \epsilon_2 \epsilon_j,$$

whence

$$\epsilon_1 = \sum_{i,j=1}^n t_{ij} \epsilon_i \epsilon_2 \epsilon_j$$

is in $\mathfrak{C}$, a contradiction. Hence $f(\xi)$ is differentiable at no $\alpha \in \mathfrak{A}$.

Suppose that $\mathfrak{A}$ is not normal. Then $\exists$ an element $\epsilon_2 \in \mathfrak{A}$, independent of ϵ_1 , lying in the centrum of $\mathfrak{A}$. Consider again $f(\xi) = x_2 \epsilon_1$ and suppose that $f(\xi)$ is differentiable at $\xi = \alpha$. Choosing $\delta = d_2 \epsilon_2$, then as above

$$(3.5) \quad \epsilon_1 = \sum_{i,j=1}^n t_{ij} \epsilon_i \epsilon_2 \epsilon_j = \left(\sum_{i,j=1}^n t_{ij} \epsilon_i \epsilon_j \right) \epsilon_2.$$

Since $\lim_{\delta \rightarrow 0} \sum_{i,j=1}^n t_{ij} \epsilon_i \epsilon_j$ exists, 3.5 implies that $\epsilon_1 = f'(\alpha) \epsilon_2$. But by Theorem 3.1, $f'(\alpha) = \partial f / \partial x_1|_{\xi=\alpha} = 0$. Hence $\epsilon_1 = 0$, a contradiction.

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