TANGENTIAL LIMITS OF FUNCTIONS OF THE CLASS S_α

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1. Introduction. The class $S_\alpha = \{f(z) = \sum_{n=0}^{\infty} c_n z^n \mid \sum n^\alpha |c_n|^2 < \infty\}$ has been discussed by Salem and Zygmund [6; 7], Broman [1] and Carleson [2]. It has been shown [1; 2] that $f(z)$ has nontangential limits at all points of the unit circle except possibly for a set whose capacity of order $1-\alpha$ is zero. Frostman [4] shows the existence of nontangential limits for Blaschke products, except in a set whose capacity of order α is zero, if the zeros $a_i, i=1, 2, \dots$, of the product satisfy the condition $\sum (1-|a_i|)^\alpha < \infty$. He proves a similar theorem for their derivatives. Cargo [3] shows that for $1 < \gamma < 1/\alpha$, the Blaschke products discussed by Frostman have limits along paths meeting the circle with order of tangency $\gamma-1$, except for sets whose capacity of order $\alpha\gamma$ is zero. He proves analogous results for the successive derivatives. His conjecture that similar results hold for S_α motivates this note.

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2. Definitions and results. Tangential limits are defined as follows: Let $R[\tau, \psi] = \{z \mid 1-|z| \geq |z-e^{i\psi}|^\tau, z \neq e^{i\psi}\}$. If $z_j \rightarrow e^{i\psi}, z_j \in R[\tau, \psi], j=1, 2, \dots$ implies that $\lim_{j \rightarrow \infty} f(z_j)$ exists, we say that $\tau - \lim_{z \rightarrow e^{i\psi}} f(z) = \lim_{j \rightarrow \infty} f(z_j)$.

A set D has positive capacity of order $\beta, \beta < 1$, if there exists a positive distribution of unit mass on D, μ , for which $\sup_{z_0} \int |z-z_0|^{-\beta} d\mu(z)$ is finite. Otherwise $\text{cap}_\beta D = 0$.

Following Zygmund [7, p. 139], we let

$$f_q(z) = \sum c_n (in)^{-q} z^n, \quad f^r(z) = \sum c_n (in)^r z^n$$

be the fractional integral of order q and the fractional derivative of order r of $f(z)$.

We wish to show:

THEOREM. Let $f(z) = \sum c_n z^n, \sum n^\alpha |c_n|^2 < \infty$.

(1) If $0 < \gamma < \alpha$, there exists a $\tau - \lim_{z \rightarrow e^{i\psi}} f(z)$ for every $\tau < (1-\gamma)/(1-\alpha)$ except possibly for $e^{i\psi} \in E_\gamma^0$, where $\text{cap}_{1-\gamma} E_\gamma^0 = 0$.

(2) If $r < \alpha/2, 0 < \gamma < \alpha - 2r$, there exists a $\tau - \lim_{z \rightarrow e^{i\psi}} f^r(z)$ for every $\tau < (1-\gamma)/(1-\alpha+2r)$ except possibly for $e^{i\psi} \in E_\gamma^r$, where $\text{cap}_{1-\gamma} E_\gamma^r = 0$.

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(3) If $q < (1-\alpha)/2$, $0 < \gamma < \alpha + 2q$, there exists a τ - $\lim_{z \rightarrow e^{i\psi}} f_q(z)$ for every $\tau < (1-\gamma)/(1-\alpha-2q)$, except possibly for $e^{i\psi} \in E_\gamma^q$, where $\text{cap}_{1-\gamma} E_\gamma^q = 0$.

3. **Proof.** Let the principal branch of $(1-z)^{\beta-1} = \sum A_n(\beta) z^n$. It has been shown [6] that $A_n(\beta) = n^{-\beta} [1 + O(1/n)] / \Gamma(1-\beta)$. Let $c_n^0(\alpha/2) = c_n/A_n(\alpha/2) + O(1/n)$, $c_n^r(\alpha/2) = c_n/A_n(\alpha/2) + O^r(1/n)$, and $c_n^q(\alpha/2) = c_n/A_n(\alpha/2) + O^q(1/n)$, and let $s_0(\theta)$, $s_r(\theta)$ and $s_q(\theta)$ be functions having $\sum c_n^0(\alpha/2) e^{in\theta}$, $\sum c_n^r(\alpha/2) e^{in\theta}$, and $\sum c_n^q(\alpha/2) e^{in\theta}$ as their Fourier series. Since $\sum n^\alpha |c_n|^2 < \infty$, the $O(1/n)$, $O^r(1/n)$, $O^q(1/n)$ can be chosen so that $\sum |c_n^0(\alpha/2)|^2 < \infty$, $\sum |c_n^r(\alpha/2)|^2 < \infty$, $\sum |c_n^q(\alpha/2)|^2 < \infty$, $|s_0(\theta)|^2$, $|s_r(\theta)|^2$, $|s_q(\theta)|^2$ are integrable, and

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - ze^{-i\theta})^{\alpha/2-1} s_0(\theta) d\theta, \\ f^r(z) &= \frac{i^r}{2\pi} \frac{\Gamma(1+r-\alpha/2)}{\Gamma(1-\alpha/2)} \int_{-\pi}^{\pi} (1 - ze^{-i\theta})^{\alpha/2-r-1} s_r(\theta) d\theta, \\ f_q(z) &= \frac{i^{-q}}{2\pi} \frac{\Gamma(1-q-\alpha/2)}{\Gamma(1-\alpha/2)} \int_{-\pi}^{\pi} (1 - ze^{-i\theta})^{\alpha/2+q-1} s_q(\theta) d\theta. \end{aligned}$$

Let $E_\gamma^t = \{e^{i\psi} | \int_{-\pi}^{\pi} |e^{i\psi} - e^{i\theta}|^{\gamma-1} |s_t(\theta)|^2 d\theta = \infty\}$, $t=0, r, q$. du Plessis [5] has shown that for $|s_t(\theta)|^2$ integrable, which is true in our case, $\text{cap}_{1-\gamma} E_\gamma^t = 0$, for $\gamma < 1$.

If $\gamma < \beta < 1$, $z \in R[(1-\gamma)/(1-\beta), \psi]$, for properly chosen K , $K > 0$, $|e^{i\theta} - z| \geq \min_{s \in R[(1-\gamma)/(1-\beta), \psi]} |e^{i\theta} - s| \geq K |e^{i\theta} - e^{i\psi}|^{(1-\gamma)/(1-\beta)}$, and so

$$(1) \quad \int_{-\pi}^{\pi} |e^{i\theta} - z|^{\beta-1} |s_q(\theta)|^2 d\theta \leq K^{\beta-1} \int_{-\pi}^{\pi} |e^{i\theta} - e^{i\psi}|^{\gamma-1} |s_q(\theta)|^2 d\theta.$$

Pick $a > 0$, $\epsilon > 0$. By Schwarz's inequality

$$\begin{aligned} & \left| \int_{\psi-a}^{\psi+a} (1 - ze^{-i\theta})^{\alpha/2+q-1} s_q(\theta) d\theta \right|^2 \\ & \leq \left| \int_{\psi-a}^{\psi+a} |e^{i\theta} - z|^{\alpha+2q-1-\epsilon} |s_q(\theta)|^2 d\theta \right| \cdot \left| \int_{\psi-a}^{\psi+a} |e^{i\theta} - z|^{\epsilon-1} d\theta \right|. \end{aligned}$$

Choose $\tau = (1-\gamma)/(1+\epsilon-\alpha-2q)$, $z \in R[\tau, \psi]$ and use (1) to obtain

$$\begin{aligned} & \left| \int_{\psi-a}^{\psi+a} (1 - ze^{-i\theta})^{\alpha/2+q-1} s_q(\theta) d\theta \right|^2 \\ & \leq K^{\alpha+2q-\epsilon-1} \left| \int_{-\pi}^{\pi} |e^{i\theta} - e^{i\psi}|^{\gamma-1} |s_q(\theta)|^2 d\theta \right| \cdot \left| \int_{\psi-a}^{\psi+a} |e^{i\theta} - z|^{\epsilon-1} d\theta \right|. \end{aligned}$$

Since the second factor on the right goes to zero with a , for $e^{i\psi} \notin E_\gamma^a$, uniformly for $z \in R[\tau, \psi]$,

$$(2) \quad \lim_{a \rightarrow 0} \int_{\psi-a}^{\psi+a} (1 - ze^{-i\theta})^{\alpha/2+\alpha-1} s_a(\theta) d\theta = 0.$$

Since $\int_{-\pi}^{\psi-a} + \int_{\psi+a}^{\pi} (1 - ze^{-i\theta})^{\alpha/2+\alpha-1} s_a(\theta) d\theta$ is analytic at $e^{i\psi}$, (2) implies the existence at $e^{i\psi}$ of

$$\tau = \lim_{z \rightarrow e^{i\psi}} \int_{-\pi}^{\pi} (1 - ze^{-i\theta})^{\alpha/2+\alpha-1} s_a(\theta) d\theta.$$

This proves (3). The proof for (1) and (2) is the same.

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