A SEQUENCE OF INTEGERS RELATED TO THE BESSEL FUNCTIONS

L. CARLITZ¹

Let $j_{\nu,r}$ denote the zeros of $z^{-\nu}J_{\nu}(z)$, where $J_{\nu}(z)$ is the Bessel function of the first kind, and put

$$(1) \quad \sigma_{2n}(\nu) = \sum_{r=1}^{\infty} (j_{\nu,r})^{-2n} \quad (n = 1, 2, 3, \dots).$$

Properties of $\sigma_{2n}(\nu)$ have been discussed in a recent paper by Kishore [2]. We remark that $\sigma_{2n}(\nu)$ is a rational function of ν with rational coefficients; the first twelve functions have been computed by Lehmer [3].

For $\nu = \pm \frac{1}{2}$, $\sigma_{2n}(\nu)$ is expressible in terms of the numbers of Bernoulli and Genocchi by means of the following formulas:

$$\begin{aligned} \sigma_{2n}(\tfrac{1}{2}) &= (-1)^{n-1} \frac{2^{2n-1}}{(2n)!} B_{2n}, \\ \sigma_{2n}(-\tfrac{1}{2}) &= (-1)^n \frac{2^{2n-2}}{(2n)!} G_{2n}, \end{aligned}$$

where

$$B^n = (B + 1)^n \quad (n \neq 1), \quad G_n = 2(1 - 2^n)B_n.$$

In view of the known arithmetic properties of these and related numbers, it is of some interest to look for arithmetic properties of $\sigma_{2n}(\nu)$ for other values of ν . In the present note we consider the case $\nu = 0$. Elsewhere [1] the writer has discussed the coefficients of $(J_0(\nu))^{-1}$.

It will be convenient to define

$$(2) \quad a_r = 2^{2r} r! (r-1)! \sigma_{2r}(0) \quad (r \geq 1).$$

Thus the formulas [2, (14), (22)]

$$\begin{aligned} \sum_{r=1}^n (-1)^r 2^{2r} (r!)^2 \binom{n}{r} \binom{\nu+n}{r} \sigma_{2r}(\nu) + n &= 0, \\ (\nu+n) \sigma_{2n}(\nu) &= \sum_{r=1}^{n-1} \sigma_{2r}(\nu) \sigma_{2n-2r}(\nu) \end{aligned}$$

reduce to

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$$(3) \quad \sum_{r=1}^n (-1)^r \binom{n}{r} \binom{n-1}{r-1} a_r + 1 = 0 \quad (n \geq 1),$$

$$(4) \quad a_{n+1} = \sum_{r=1}^n \binom{n}{r} \binom{n}{r-1} a_r a_{n-r+1} \quad (n \geq 1),$$

respectively.

Since $a_1 = 1$ we find, using either (3) or (4), that

$$a_2 = 1,$$

$$a_3 = 2^2,$$

$$a_4 = 3 \cdot 11,$$

$$a_5 = 2^3 \cdot 3 \cdot 19,$$

$$a_6 = 2^2 \cdot 5 \cdot 11 \cdot 43,$$

$$a_7 = 2^4 \cdot 3 \cdot 5^2 \cdot 229,$$

$$a_8 = 3 \cdot 5 \cdot 7 \cdot 167 \cdot 607.$$

It is evident from (4) that the a_n are positive integers. If $n = p$, a prime, it follows from (3) that

$$(5) \quad a_p \equiv 1 \pmod{p}.$$

This result can be extended. We recall that if [4]

$$n = n_0 + n_1 p + n_2 p^2 + \cdots \quad (0 \leq n_j < p),$$

$$r = r_0 + r_1 p + r_2 p^2 + \cdots \quad (0 \leq r_j < p)$$

then

$$(6) \quad \binom{n}{r} \equiv \binom{n_0}{r_0} \binom{n_1}{r_1} \binom{n_2}{r_2} \cdots \pmod{p}.$$

In particular (6) implies

$$\binom{mp}{rp} \equiv \binom{m}{r}, \quad \binom{mp-1}{rp-1} \equiv \binom{m-1}{r-1} \pmod{p}$$

and

$$\binom{mp}{r} \equiv 0 \pmod{p} \quad (p \nmid r).$$

Thus if $n = mp$, (3) becomes

$$(7) \quad \sum_{r=1}^m (-1)^r \binom{m}{r} \binom{m-1}{r-1} a_{rp} + 1 \equiv 0 \pmod{p}.$$

Comparing (7) with (3) it is evident that

$$(8) \quad a_{mp} \equiv a_m \pmod{p}$$

and somewhat more generally

$$(9) \quad a_{m p^r} \equiv a_m \pmod{p} \quad (r = 1, 2, 3, \dots).$$

We show next that

$$(10) \quad a_n \equiv 0 \pmod{p} \quad (p < n < 2p).$$

For $n = p + 1$, this is obvious from (4). Assuming that (10) holds up to and including the value n , (4) implies

$$(11) \quad a_{n+1} \equiv \sum_{r=n-p+1}^p \binom{n}{r} \binom{n}{r-1} a_r a_{n-r+1} \pmod{p}.$$

If $n+1 = p+m$, where $1 \leq m < p$, then by (6)

$$\begin{aligned} \binom{n}{r} &= \binom{p+m-1}{r} \equiv \binom{m-1}{r} \equiv 0 \pmod{p} \quad (m \leq r < p), \\ \binom{n}{p-1} &= \binom{p+m-1}{p-1} \equiv \binom{m-1}{p-1} \equiv 0 \pmod{p}. \end{aligned}$$

Thus (11) reduces to $a_{n+1} \equiv 0 \pmod{p}$. This completes the proof of (10).

It is now easy to prove the more general congruence

$$(12) \quad a_n \equiv 0 \pmod{p} \quad (n > p, p \nmid n)$$

by induction. Indeed if

$$n+1 = kp + m, \quad 1 \leq m < p,$$

then by (6) and the inductive hypothesis

$$a_{n+1} \equiv 2 \binom{n}{m} \binom{n}{m-1} a_m a_{kp} \pmod{p};$$

but by (6)

$$\binom{n}{m} = \binom{kp+m-1}{m} \equiv \binom{m-1}{m} \equiv 0,$$

so that $a_{n+1} \equiv 0 \pmod{p}$.

We remark that

$$(13) \quad a_n \equiv 0 \pmod{n-1} \quad (n > 1).$$

Indeed if we put

$$a_n = (n-1)b_n \quad (n > 1),$$

(4) becomes for $n > 1$

$$\begin{aligned} nb_{n+1} &= 2n(n-1)b_n + \sum_{r=2}^{n-1} \binom{n}{r} \binom{n}{r-1} (n-r)(r-1)b_r b_{n-r+1} \\ &= 2n(n-1)b_n + n^2 \sum_{r=2}^{n-1} \binom{n-1}{r} \binom{n-1}{r-2} b_r b_{n-r+1}, \end{aligned}$$

so that

$$(14) \quad b_{n+1} = 2(n-1)b_n + n \sum_{r=2}^{n-1} \binom{n-1}{r} \binom{n-1}{r-2} b_r b_{n-r+1} \quad (n > 1).$$

Since $b_2 = 1$ it is evident from (14) that b_n is integral for all $n \geq 2$. This proves (13).

Returning to (4) it is clear that

$$a_{p+1} \equiv 2pa_p \pmod{p^2}.$$

Combining this with (5) we get

$$(15) \quad a_{p+1} \equiv 2p \pmod{p^2}.$$

Similarly we have for $p > 2$

$$a_{p+2} \equiv 2a_{p+1} + 2 \binom{p+1}{2} (p+1)a_p \pmod{p^2}.$$

Using (5) and (15) this reduces to

$$(16) \quad a_{p+2} \equiv 5p \pmod{p^2}.$$

In the same way if $p > 3$ we get

$$\begin{aligned} a_{p+3} &\equiv 2a_{p+2} + 2 \binom{p+2}{2} (p+2)a_{p+1} \\ &\quad + 8 \binom{p+2}{3} \binom{p+2}{2} a_p \pmod{p^2}, \end{aligned}$$

which reduces to

$$(17) \quad a_{p+3} \equiv \frac{92}{3} p \pmod{p^2} \quad (p > 3).$$

To get a general result of this kind we put

$$(18) \quad a_{p+n} \equiv p c_n^{(p)} \pmod{p^2} \quad (0 < n < p),$$

so that by (10) $c_n^{(p)}$ is integral. Now by (4)

$$\begin{aligned} a_{p+n} &= \sum_{r=1}^{p+n-1} \binom{p+n-1}{r} \binom{p+n-1}{r-1} a_r a_{p+n-r} \\ &\equiv 2 \sum_{r=1}^n \binom{p+n-1}{r} \binom{p+n-1}{r-1} a_r a_{p+n-r} \pmod{p^2}. \end{aligned}$$

Making use of (18) and (6) this becomes

$$(19) \quad c_n^{(p)} \equiv 2 \sum_{r=1}^{n-1} \binom{n-1}{r} \binom{n-1}{r-1} a_r c_{n-r}^{(p)} + \frac{2}{n} a_n \pmod{p} \quad (1 < n < p).$$

By means of (19) the $c_n^{(p)}$ can be computed. However, it is simpler to define a single sequence $\{c_n\}$ by means of

$$(20) \quad c_n = 2 \sum_{r=1}^{n-1} \binom{n-1}{r} \binom{n-1}{r-1} a_r c_{n-r} + \frac{2}{n} a_n \quad (n > 1)$$

with $c_1 = 2$. We evidently have

$$(21) \quad c_n^{(p)} \equiv c_n \pmod{p} \quad (0 < n < p).$$

The c_n as defined by (20) are not integral. If we put

$$(22) \quad c'_n = n c_n \quad (n \geq 1), \quad c'_0 = 1,$$

then (20) becomes

$$(23) \quad c'_n = 2 \sum_{r=1}^n \binom{n}{r} \binom{n-1}{r-1} a'_r c'_{n-r} \quad (n \geq 1);$$

the c'_n are therefore integral. Moreover it follows from (23) that

$$\sum_{n=1}^{\infty} \frac{c'_n \left(\frac{x}{2}\right)^{2n}}{n!(n-1)!} = 2 \sum_{r=1}^{\infty} \frac{a_r \left(\frac{x}{2}\right)^{2n}}{r!(r-1)!} \sum_{n=0}^{\infty} \frac{c'_n \left(\frac{x}{2}\right)^{2n}}{n!n!}.$$

If we put

$$C(x) = \sum_{n=0}^{\infty} \frac{c'_n \left(\frac{x}{2}\right)^{2n}}{n!n!}, \quad A(x) = \sum_{r=1}^{\infty} \frac{a_r \left(\frac{x}{2}\right)^{2n}}{r!(r-1)!}$$

then

$$\frac{x}{2} C'(x) = \sum_{n=1}^{\infty} \frac{c_n' \left(\frac{x}{2}\right)^{2n}}{n!(n-1)!},$$

so that

$$\frac{x}{2} C'(x) = 2A(x)C(x).$$

But [2]

$$A(x) = \sum_{r=1}^{\infty} \sigma_{2r}(0)x^{2r} = -\frac{1}{2}x \frac{J_0'(x)}{J_0(x)}.$$

It follows that

$$\frac{C'(x)}{C(x)} = -\frac{J_0'(x)}{J_0(x)},$$

which yields

$$(24) \quad C(x) = (J_0(x))^{-2}.$$

We have accordingly found a simple generating function for the c_n .

It follows from (13) that if $n \equiv 1 \pmod{p^k}$ then $a_n \equiv 1 \pmod{p^k}$. Using (4) and (8) it is not difficult to show that if $n = mp^k + 1$ then

$$(25) \quad a_n \equiv mp^k a_m \pmod{p^{k+1}}.$$

We shall now show that if

$$(26) \quad n = n_s p^s + n_{s+1} p^{s+1} + \cdots + n_t p^t \quad (0 \leq n_j < p)$$

and $n_s \geq 1$, $n_t \geq 1$, then

$$(27) \quad a_n \equiv 0 \pmod{p^{t-s}}.$$

The proof is by induction on n . We use (4) with n replaced by $n-1$. Let $1 \leq r < n$ and put

$$r = r_s p^{s'} + r_{s'+1} p^{s'+1} + \cdots + r_{t'} p^{t'} \quad (0 \leq r_j < p),$$

$$m = n - r = m_{s''} p^{s''} + m_{s''+1} p^{s''+1} + \cdots + m_{t''} p^{t''} \quad (0 \leq m_j < p).$$

Clearly either $s' \leq s$ or $s'' \leq s$; also $t' \leq t$ and $t'' \leq t$. If $t' = t'' = t$ there is evidently nothing to prove.

(i) $t' = t$, $t'' < t$. By the inductive hypothesis

$$a_r \equiv 0 \pmod{p^{t'-s'}}, \quad a_{n-r} \equiv 0 \pmod{p^{t''-s''}},$$

so that

$$a_r a_{n-r} \equiv 0 \pmod{p^{t'+t''-s'-s''}}.$$

If $s' \leq t''$ it follows (since $t'' \geq \max(s', s'')$) that $t' + t'' - s' - s'' \geq t - s$. If however $s' > t''$ we examine the binomial coefficient $C_{n-1, n}$.

We recall that if

$$\begin{aligned} n &= n_0 + n_1 p + \cdots + n_k p^k & (0 \leq n_j < p), \\ S(n) &= n_0 + n_1 + \cdots + n_k, \end{aligned}$$

then $n!$ is divisible by exactly p^e , where

$$(p-1)^e = n - S(n).$$

It follows that $\binom{n-1}{r}$ is divisible by $p^{s'-s''}$. Since $(t' + t'' - s' - s'') + (s' - s'') = t + t'' - 2s \geq t - s$ we get

$$\binom{n-1}{r} a_r a_{n-r} \equiv 0 \pmod{p^{t-s}}.$$

(ii) $t' < t$, $t'' < t$. We may suppose that $t'' \leq t' = t - 1$. Also it is clear that $s' \leq t''$. Then

$$t' + t'' - s' - s'' \geq t - s - 1;$$

indeed if $s = \min(s', s'')$ we get

$$t' + t'' - s' - s'' \geq t - s.$$

Thus only the case $s = \min(s', s'')$ requires further examination. With the present hypothesis we evidently have

$$\binom{n}{r} \equiv 0 \pmod{p};$$

but when $s = \min(s', s'')$ then either n and r or n and $n - r$ are divisible by the same power of p . It follows that either

$$\binom{n-1}{r} \quad \text{or} \quad \binom{n-1}{r-1}$$

is divisible by p . Consequently

$$\binom{n-1}{r} \binom{n-1}{r-1} a_r a_{n-r} \equiv 0 \pmod{p^{t-s}}.$$

This completes the proof of (26).

SUMMARY. The sequence of positive integers $\{a_n\}$ defined by (2)—or alternatively by (3) or (4)—have the following properties.

1. $a_{mp} \equiv a_m, \quad a_p \equiv 1 \pmod{p}.$
2. $a_n \equiv 0 \pmod{p} \quad (n > p, p + n).$
3. $a_n \equiv 0 \pmod{n-1} \quad (n > 1).$
4. $a_{p+n} \equiv c_n p \pmod{p^2} \quad (1 \leq n < p),$

where the c_n are defined by

$$1 + \sum_{n=1}^{\infty} \frac{c_n \left(\frac{x}{2}\right)^{2n}}{(n-1)!(n-1)!} = (J_0(x))^{-2};$$

moreover nc_n is integral.

5. If $n = mp^k + 1$ then

$$a_n \equiv mp^k a_m \pmod{p^{k+1}}.$$

6. If $p^s | n, p^{s+1} \nmid n, p^t \leq n < p^{t+1}$ then

$$a_n \equiv 0 \pmod{p^{t-s}}.$$

The following values of a_n were computed by R. Carlitz in the Duke University Computing Laboratory.

$$\begin{aligned} a_9 &= 2^4 \cdot 5 \cdot 7 \cdot 19 \cdot 31 \cdot 1607 \\ a_{10} &= 2^3 \cdot 3^2 \cdot 7 \cdot 199 \cdot 328981 \\ a_{11} &= 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 83 \cdot 3000553 \\ a_{12} &= 2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 2389 \cdot 4153159 \\ a_{13} &= 2^5 \cdot 3^2 \cdot 5 \cdot 7^4 \cdot 11 \cdot 29 \cdot 97 \cdot 139 \cdot 1663 \\ a_{14} &= 2^4 \cdot 3^2 \cdot 5 \cdot 11 \cdot 13 \cdot 37 \cdot 107 \cdot 1283 \cdot 5952613 \\ a_{15} &= 2^6 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 563 \cdot 797 \cdot 227966279 \\ a_{16} &= 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 41 \cdot 2390700514417253 \\ a_{17} &= 2^5 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 461 \cdot 1342361 \cdot 33327739 \\ a_{18} &= 2^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 79 \cdot 199729 \cdot 139135943558279 \\ a_{19} &= 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot a'_{19} \\ a_{20} &= 2^3 \cdot 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 1109 \cdot a'_{20} \\ a_{21} &= 2^6 \cdot 3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 47 \cdot a'_{21} \\ a_{22} &= 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot a'_{22} \\ a_{23} &= 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot a'_{23} \\ a_{24} &= 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot a'_{24} \end{aligned}$$

$$a_{25} = 2^6 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot a'_{25}$$

$$a_{26} = 2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 23^2 \cdot 79 \cdot a'_{26}$$

$$a_{27} = 2^7 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 43^3 \cdot a'_{27}$$

$$a_{28} = 2^4 \cdot 3^3 \cdot 5^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot a'_{28}$$

$$a_{29} = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot a'_{29}$$

$$a_{30} = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot a'_{30}$$

The numbers $a'_{19}, \dots, a'_{30}$ have not been factored completely but at any rate have no prime divisors $< 10^4$. The number a'_{30} has 47 digits.

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DUKE UNIVERSITY