A FURTHER THEOREM OF THE SYLVESTER TYPE

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1. **Introduction.** In [2] the following theorem is established:

THEOREM I. *If $\{S_i\}$ is a finite collection of disjoint, bounded, closed sets in E_n at least one of which is infinite, and not all of which are subsets of the same line, then there exists a line cutting exactly two of the sets.*

On the other hand Edelstein [1] proved:

THEOREM II. *Let $\{S_i\}$ be a finite collection of sets in E_n , nonempty, mutually exclusive, closed and bounded and not all subsets of the same line. Let every set of the collection have a finite number of components and let at least one of the sets of $\{S_i\}$ have a component which is not a single point. Then there exists a hyperplane cutting exactly two of the sets.*

Theorem II has both substantially stronger hypotheses and a stronger conclusion. The present paper obtains the stronger conclusion of Theorem II from the weaker hypotheses of Theorem I. For a more elaborate bibliography and historical comment on problems of the Sylvester type see [2], [3], and [4].

Precisely what the situation is when the requirement that one of the sets be infinite is dropped is not known. (See [2, Theorem 4.1].) This matter is still under investigation.

2. **Admissible directions.** We consider the euclidean n -space E_n . Any $(n-1)$ -dimensional linear subspace of E_n will be referred to as a *hyperplane*, the surface $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ as the *unit hypersphere*.

If Σ is any compact point set in E_n there corresponds to each direction γ (realized as a unit vector or as a point on the unit hypersphere) a unique support hyperplane π of Σ , which is (a) perpendicular to γ , and (b) such that the perpendicular direction from π to any point of Σ which is not on π is the same as (rather than opposite to) the direction of γ . The correspondence between all directions γ and all support hyperplanes π of Σ is in general bi-unique. Only when all of Σ lies in one hyperplane of E_n there exists at least one pair of opposite directions to which the same support hyperplane π corresponds.

Let $\Sigma \equiv \bigcup_{i=1}^p S_i$, where each S_i is compact and $S_i \cap S_j = \emptyset$ for $i \neq j$. We shall call a direction γ an *admissible direction* (with respect to the partition $\Sigma \equiv S_1 \cup S_2 \cup \cdots \cup S_p$) if the corresponding support hyper-

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plane π of Σ intersects only one of the sets S_i . If π intersects more than one of these sets we call γ an *inadmissible direction*.

LEMMA 1. *The set of all admissible directions with respect to the partition $\Sigma \equiv \bigcup_{i=1}^p S_i$ is open and everywhere dense on the unit hypersphere.*

PROOF. Let γ_0 be an admissible direction. Let π_0 be the support hyperplane of Σ which corresponds to γ_0 and let π_0 intersect S_k . Then all points of $\Sigma - S_k$ lie on one side of π_0 but not on π_0 . Let P_k be an arbitrary point of $\pi_0 \cap S_k$. The line connecting any point of $\Sigma - S_k$ with P_k forms an angle α with π_0 , where $0 < \alpha \leq 90^\circ$. If α_0 is the minimum of all these values α then $\alpha_0 > 0$, and it is clear that any direction γ making an angle less than α_0 with γ_0 will also be an admissible direction. The set of admissible directions is therefore open.

It remains to show that in any neighborhood of a given direction γ_1 there is an admissible direction. It suffices to assume that γ_1 is an inadmissible direction. Let π_1 be the support hyperplane of Σ corresponding to γ_1 ; the set $\pi_1 \cap \Sigma$ then contains points from more than one of the sets S_i . Let q be an $(n-2)$ -dimensional linear variety in π_1 which is a support for $\pi_1 \cap \Sigma$ and intersects $\pi_1 \cap \Sigma$ in a single point, say $P_j \in S_j$. (Such a support q can be obtained, for instance, by choosing an arbitrary point O in π_1 , $O \notin \pi_1 \cap \Sigma$. Let P_j be a point of $\pi_1 \cap \Sigma$ at maximal distance from O , and let q be perpendicular to the line OP_j and pass through the point P_j .) If now π_1 is rotated about q in the proper direction through a sufficiently small angle, the resulting hyperplane will intersect S_j only and, therefore, will be parallel to a support hyperplane π which intersects S_j only. The direction γ corresponding to π is admissible and arbitrarily close to γ_1 .

3. The main theorem. For the proof of the main theorem, we need the following Lemma 2, in which the notion of a pseudo-support is used. This notion was introduced in [2, p. 328] and will be repeated here for the sake of completeness.

DEFINITION. A hyperplane in E_n which intersects a point-set S and is such that at least one of the open half-spaces defined by that hyperplane contains at most a finite number of points of S is a pseudo-support of S .

LEMMA 2. *If $\{S_i\}$ is a finite collection of two or more mutually disjoint compact sets in E_n , $n \geq 2$, with $\Sigma = \bigcup S_i$ infinite and if none of the sets $\Sigma - S_i$ lies in a hyperplane, then there exist an index k , a point $P_k \in S'_k$, the derived set of S_k , an $(n-2)$ -dimensional linear variety l through P_k , and a hyperplane π containing l with the following properties:*

- (i) π is a pseudo-support for $\Sigma - S_k$;
- (ii) l contains no points of Σ other than P_k ;
- (iii) at least one of the half-spaces of π bounded by l is free of points of $(\Sigma - S_k)'$;
- (iv) there exists a sequence of points $\{Q_r\}$ with $Q_r \in S_k$, $Q_r \notin \pi$ and $\lim Q_r = P_k$.

PROOF. We begin by choosing a support hyperplane σ of Σ' which intersects Σ' in a single point, say $P_k \in S_k$. Consider now an arbitrary $(n-2)$ -dimensional linear variety l in σ through P_k , subject only to the restriction that l should contain no point of Σ other than P_k . This is evidently possible since there are at most denumerably many points of Σ on σ . We remark that this restriction on the choice of l will, in the procedure which follows, ensure that in any consideration of the approach of P_k by points of S_k , the possibility that such an approach might be possible only along l is certainly eliminated.

We now distinguish two cases:

Case (a). There are points of $\Sigma - S_k$ on σ . If P_k can be approached by points of S_k not on σ then σ has all desired properties, and we take $\pi = \sigma$. If P_k can be approached only by points of S_k on σ then rotate σ about l in either direction. By the hypothesis of the lemma there must be points of $\Sigma - S_k$ not on σ . Hence if we denote by π the first hyperplane obtained in the rotation that intersects $\Sigma - S_k$, then $\pi \neq \sigma$ and π will satisfy all the requirements of the lemma.

Case (b). There are no points of $\Sigma - S_k$ on σ . Rotate σ about l in either direction. Let τ be the first hyperplane obtained in this way that intersects $\Sigma - S_k$. If P_k can be approached by points of S_k not on τ , then take $\pi = \tau$. If, however, P_k can only be approached by points of S_k on τ , then return to σ and rotate it in the opposite direction about l . Let the first hyperplane that intersects $\Sigma - S_k$ be π . Since not all of $\Sigma - S_k$ can lie on τ , we must have $\pi \neq \tau$, and π will now have the properties desired in the lemma.

THEOREM III. Let $\{S_i\}$ be a finite collection of three or more mutually disjoint compact sets in E_n with $\Sigma = \cup S_i$ infinite. Then either Σ is a subset of a line or there exists a hyperplane intersecting exactly two of the sets S_i .

PROOF. The case $n=2$ is taken care of by [2, Theorem 2.1]. Assume the theorem valid for E_{n-1} ($n \geq 3$) and let Σ be a nonlinear set. We first dispose of the case in which one of the sets $\Sigma - S_i$, say $\Sigma - S_k$, lies in some hyperplane ρ , in which case Lemma 2 does not apply. If S_k also lies in ρ then $\Sigma \subset \rho$ and (by induction) an $(n-2)$ -dimensional linear variety λ in ρ exists which intersects exactly two of the sets

S_i . Hence any hyperplane (in E_n) containing λ and not containing ρ will do the same.

If S_h has points not on ρ , let q be an $(n-2)$ -dimensional linear variety in ρ which intersects exactly one of the sets of $\Sigma - S_h$. (For instance let q be a support, relative to ρ as a containing space, of $\Sigma - S_h$, intersecting $\Sigma - S_h$ in a single point.) Now any hyperplane through q and a point of S_h not on ρ will intersect exactly two of the sets S_i .

We thus assume that none of the sets $\Sigma - S_i$ lies in a hyperplane and apply Lemma 2. Let then π and l satisfy Lemma 2. We think of the hyperplane π as "horizontal" and assume that there are at most finitely many points of $\Sigma - S_k$ "above" π . Furthermore, the $(n-2)$ -dimensional linear variety l divides π into two half-hyperplanes of which the "left" one, say, will be assumed to be free of points of $(\Sigma - S_k)'$.

Under the assumption that both directions perpendicular to l in π are admissible directions (in the sense of §2) with respect to each of the following partitions:

$$\begin{aligned}\pi \cap (\Sigma - S_k) &= \bigcup_{i \neq k} (\pi \cap S_i), \\ \pi_r \cap (\Sigma - S_k) &= \bigcup_{i \neq k} (\pi_r \cap S_i),\end{aligned}$$

where π_r denotes the open right half-hyperplane of π bounded by l , we denote l by l^* . (If $\pi_r \cap (\Sigma - S_k) = \emptyset$ the second of the above partitions should, of course, be disregarded.) If l does not satisfy these conditions we consider a new $(n-2)$ -dimensional linear variety l^* chosen as follows.

On the unit hypersphere in π we surround the two ends γ_1 and γ_2 of the diameter, the direction of which is perpendicular to l , by neighborhoods N_1 and N_2 respectively.

If N_1 and N_2 are sufficiently small then, clearly, any hyperplane through P_k and perpendicular to $\gamma_3 \in N_1 \cup N_2$ will be disjoint from $\Sigma - S_k$.

Lemma 1, together with the obvious fact that the intersection of finitely many sets, each of which is open and everywhere dense on the unit hypersphere is itself everywhere dense, ensures that a direction $\gamma^* \in N_1 \cup N_2$ exists such that γ^* together with the direction opposite to γ^* are both admissible with respect to the above two partitions. We take l^* perpendicular to γ^* and passing through P_k .

It must be noted, however, that this variety l^* , if $l^* \neq l$, does not

necessarily satisfy any more the condition (ii) of Lemma 2. But l^* contains no points of $\Sigma - S_k$, which is enough for our purposes.

We now distinguish between two cases (not mutually exclusive).

Case (a). There exists a sequence of points $\{Q_n\}$ with $Q_n \in S_k$, $\lim Q_n = P_k$ and Q_n above π . Let m be the support, relative to π , of $\pi \cap (\Sigma - S_k)$ which is parallel to l^* and such that all of $\pi \cap (\Sigma - S_k)$ lies on the same side of m as l^* . (There are either two such supports or only one, depending on whether there are points of $\pi \cap (\Sigma - S_k)$ on both sides of l^* or not.)

The support m intersects exactly one of the sets, say S_j , $j \neq k$, and it is clear that, for sufficiently large n , the hyperplane through m and Q_n intersects S_j and S_k but no other S_i .

Case (b). There exists a sequence of points $\{Q_n\}$ with $Q_n \in S_k$, $\lim Q_n = P_k$ and Q_n below π .

(i) If $\pi_r \cap (\Sigma - S_k) \neq \emptyset$ let m be the support of $\pi_r \cap (\Sigma - S_k)$ which is parallel to l^* and such that no point of $\pi_r \cap (\Sigma - S_k)$ lies between m and l^* . As above, let m intersect the set S_j , $j \neq k$. Since there are no points of $(\Sigma - S_k)'$ to the left of l^* on π , we can again conclude that, for large n , the hyperplane through m and Q_n intersects S_j and S_k but no other S_i .

(ii) If $\pi_r \cap (\Sigma - S_k) = \emptyset$ let m be any $(n-2)$ -dimensional linear variety in π parallel to l^* , intersecting exactly one set S_j , $j \neq k$; for instance, let m be a support of $\pi \cap (\Sigma - S_k)$ parallel to l^* . In this case there are no points of $(\Sigma - S_k)'$ on π and hence, for large n , the hyperplane through m and Q_n intersects S_j and S_k but not S_i , $i \neq j$, $i \neq k$.

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