NOTE ON A PAPER BY SHEPPERD ON THE BRAID GROUP

SEYMOUR LIPSCHUTZ¹

1. **Introduction.** Let $F_n = \langle x_1, x_2, \dots, x_n \rangle$ be the free group on which the braid group of n strings, B_n , acts as a group of automorphisms. If α is a braid in the subgroup A_n of B_n where

$$B_n/A_n \cong \Sigma_n = \text{symmetric group of order } n!,$$

that is, the subgroup with identity permutation, then (see [1]), as an automorphism of F_n ,

$$\alpha(x_i) = T_i(x_\lambda)x_iT_i(x_\lambda)^{-1} \quad (i = 1, 2, \dots, n),$$

$$\alpha(x_1x_2 \dots x_n) = x_1x_2 \dots x_n.$$

Let Q_n be the subgroup of A_n which is generated by the braids

$$R_i = (\sigma_1\sigma_2 \dots \sigma_{i-2})^{i-1}(\sigma_i\sigma_{i+1} \dots \sigma_n)^{i-n} \quad (i = 2, \dots, n+1),$$

where the σ_i are the usual generators of B_n . (See [1; 4].)

A textile manufacturer asked the following question: Can one decide whether or not a braid in A_n is also in Q_n ? (It is possible to "weave" a braid in Q_n while keeping its ends "tied together.") Shepperd [6] solved the above decision problem using the generators and relations of the braid group and subgroups. In this paper we give another distinct solution working in terms of automorphisms of free groups. In particular, we use formulas developed in [3; 4].

The author wishes to thank his teacher, and friend, Professor Wilhelm Magnus who has contributed more than anyone else towards the author's knowledge of mathematics.

2. **The algorithm.** To decide whether or not α in A_n is also in Q_n , we present the following:

ALGORITHM. Rewrite α as the automorphism

$$(1) \quad \alpha(b_i) = W_i(b_\lambda) \quad (i = 1, 2, \dots, n),$$

where $b_i = x_ix_{i+1} \dots x_n$. Set $b_1 = I$, the identity element, in (1) and freely reduce. Since the group B_1 generated by b_1 is characteristic under A_n , α now induces an automorphism α^* of the free group

---

Received by the editors March 8, 1962.

¹ Work on this paper was partially sponsored by a grant from the National Science Foundation, NSF-G9659 during the summer of 1961 at the Courant Institute of Mathematics.

$F_{n-1}^* = F_n/B_1$. We have two possibilities:

(a) The automorphism α^* is not of the form

$$(2) \quad \alpha^*(b_i) = V(b_2, \dots, b_n)b_iV(b_2, \dots, b_n)^{-1} \quad (i = 2, \dots, n),$$

that is, an inner automorphism of F_{n-1}^* . Then α is not in Q_n .

(b) The automorphism α^* is of the form (2). We now see if

$$(3) \quad b_1^t \alpha(b_i) b_1^{-t} = V(R_2, \dots, R_n)(b_i) \quad (i = 1, \dots, n),$$

for any t where $|t| < T$. Here T is equal to the maximum length of $\alpha(b_i)$ plus the maximum length of $V(R_2, \dots, R_n)(b_i)$. (Notice that $V(R_2, \dots, R_n)$ is an automorphism acting on b_i .)

CASE (a). If there is no t for which (3) is true then α is not in Q_n .

CASE (b). If there is a t for which (3) is true then α is in Q_n , in fact,

$$\alpha = R_{n+1}^t V(R_2, \dots, R_n).$$

THEOREM. *The above algorithm is a solution to the decision problem of whether a braid in A_n is also in Q_n .*

PROOF. As was shown in [3; 4], the braids R_i are the following automorphisms of F_n :

$$(4) \quad R_i(b_j) = b_i a_{ij}^{-1} b_j b_i^{-1} a_{ij} \dots \quad (i = 2, \dots, n),$$

$$(5) \quad R_{n+1}(b_j) = b_1 b_j b_1^{-1},$$

where

$$a_{ij} = \begin{cases} b_1 & \text{if } j < i, \\ I, \text{ the identity,} & \text{if } i \leq j. \end{cases}$$

It is known [5] that R_{n+1} is in the center of B_n and that the other R_i are free generators of a free subgroup. Therefore, for any β in Q_n ,

$$(6) \quad \beta = R_{n+1}^t W(R_2, \dots, R_n).$$

But, by (4) and (5), the induced automorphism of β acting on F_{n-1}^* is

$$\beta^*(b_i) = W(b_2, \dots, b_n)b_iW(b_2, \dots, b_n)^{-1} \quad (i = 2, \dots, n),$$

an inner automorphism. (Another verification that the R_i , $i \neq n+1$, are free.) Thus a necessary condition for α in A_n to be in Q_n is for α^* to be an inner automorphism of F_{n-1}^* . Further, if (2) is true then, by (4) and (5),

$$\alpha \equiv V(R_2, \dots, R_n) \pmod{B_1}.$$

Now, by (5) and (6), α is in Q_n if and only if (3) is true for some t . A simple cancellation argument in (3) will show that we need not consider t if $|t| > T$.

Thus the algorithm is a solution to the above decision problem.

BIBLIOGRAPHY

1. E. Artin, *Theory of braids*, Ann. of Math. (2) **48** (1947), 101–126.
2. ———, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1925), 47–72.
3. S. Lipschutz, *On the braid group*, Ph.D. Dissertation, New York University, N. Y., 1960.
4. ———, *On a finite matrix representation of the braid group*, Arch. Math. **12** (1961), 7–12.
5. W. Magnus, *Über Automorphismen von Fundamentalgruppen berandeter Flächen*, Math. Ann. **105** (1931), 52–74.
6. J. A. H. Shepperd, *Braids which can be plaited with their threads tied together at each end*, Proc. Roy. Soc. Ser. A **265** (1962), 229–244.

NEW YORK UNIVERSITY AND
POLYTECHNIC INSTITUTE OF BROOKLYN