AN IMPROVEMENT AND GENERALIZATION OF BELLMAN-SHAPIRO'S THEOREM ON A PROBLEM IN ADDITIVE NUMBER THEORY

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1. Introduction and two important lemmas. Let $K > 1$ be a fixed positive integer. Then we can use any integer x to be uniquely represented as follows:

$$x = a_1 K^{n_1} + a_2 K^{n_2} + \cdots + a_t K^{n_t}.$$

$n_1 > n_2 > n_3 > n_4 > \cdots > n_t \geq 0$ are integers; so are $a_1, a_2, a_3, \cdots, a_t$ each of which is no larger than $K-1$. We set $\alpha(x) = \sum_{i=1}^t a_i$ and $A(x) = \sum_{y \leq x} \alpha(y)$. In the case of $K=2$, R. Bellman and H. Shapiro have proved the following relation [1]:

$$A(x) = \frac{x \log x}{2 \log 2} + O(x \log \log x).$$

In this paper we extend their result to a general case and discover a better alternative residual.

LEMMA 1.

$$A(K^n) = \frac{nK^n(K-1)}{2} + 1.$$

PROOF.

$$\begin{aligned} A(K^n) &= \sum_{x \leq K^n} \alpha(x) = \sum_{x \leq K^{n-1}} \alpha(x) + \sum_{K^{n-1} < x \leq 2K^{n-1}} \alpha(x) + \cdots \\ &\quad + \sum_{(K-1)K^{n-1} < x < K^n} \alpha(x) \\ &= \sum_{x \leq K^{n-1}} \alpha(x) + \sum_{x \leq K^{n-1}} \alpha(K^{n-1} + x) + \cdots \\ &\quad + \sum_{x \leq K^{n-1}} \alpha[(K-1)K^{n-1} + x] \\ &= (AK^{n-1}) + K^{n-1} + A(K^{n-1}) + \cdots + (K-L)K^{n-1} \\ &\quad + A(K^{n-1}) - (K-1) \\ &= KA(K^{n-1}) + K^{n-1} + 1 + \cdots + (K-1) - (K-1) \\ &= KA(K^{n-1}) + K^{n-1} \frac{K(K-1)}{2} - (K-1) \\ &= KA(K^{n-1}) + \frac{1}{2}(K-1)(K^n - 2). \end{aligned}$$

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We replace n with $n-1, n-2, \dots, 1$, and multiple $K, K^2, \dots, K^{n-1}$ in the order and again add to the above formula. Then we have (to pay attention to $A(K) = A(1) = 1$)

$$\begin{aligned} A(K^n) &= K^n + \frac{1}{2}(K-1)[nK^n - 2(1 + K + \dots + K^{n-1})] \\ &= K^n + \frac{1}{2}(K-1) \left[nK^n - 2 \frac{K^n - 1}{K - 1} \right] \\ &= \frac{1}{2}nK^n(K-1) + 1. \end{aligned}$$

LEMMA 2.

$$A(j \cdot K^n) = n \left[\frac{j \cdot K^n(K-1)}{2} \right] + \frac{j(j-1)}{2} K^n + j, \quad j \leq K-1.$$

PROOF.

$$\begin{aligned} A(j \cdot K^n) &= \sum_{x \leq j \cdot K^n} \alpha(x) = \sum_{x \leq K^n} \alpha(x) + \sum_{K^n < x \leq 2 \cdot K^n} \alpha(x) + \dots \\ &\quad + \sum_{(j-1)K^n < x \leq j \cdot K^n} \alpha(x) \\ &= \sum_{x \leq K^n} \alpha(x) + \sum_{K^n < x \leq 2K^n} \alpha(x) + \dots \\ &\quad + \sum_{x \leq K^n} \alpha[(j-1)K^n + x] \\ &= A(K^n) + K^n + A(K^n) + \dots + (j-1)K^n + A(K^n) \\ &= j \cdot A(K^n) + K^n(1 + \dots + j-1). \end{aligned}$$

We substitute the result of Lemma 1 into this and simplify in order to get the result we want.

2. Proof of Theorem 1.

THEOREM 1. For any positive integer $K \geq 2$,

$$A(x) = \frac{K-1}{2} \frac{x \log x}{\log K} + O(x).$$

The above residual is the best possible.

PROOF. Now we assume: $x = a_1 K^{n_1} + a_2 K^{n_2} + \dots + a_i K^{n_i}$, $n_1 > n_2 > \dots > n_i \geq 0$. Then,

$$\begin{aligned}
A(x) &= \sum_{y \leq x} \alpha(y) = \sum_{y \leq a_1 K^{n_1}} \alpha(y) \\
&\quad + \sum_{a_1 K^{n_1} < y \leq a_1 K^{n_1} + a_2 K^{n_2}} \alpha(y) + \cdots + \sum_{a_1 K^{n_1} + \cdots + a_{t-1} K^{n_{t-1}} < y \leq x} \alpha(y) \\
&= A(a_1 K^{n_1}) + \sum_{y \leq a_2 K^{n_2}} \alpha(a_1 K^{n_1} + y) + \cdots \\
&\quad + \sum_{y \leq a_t K^{n_t}} \alpha(a_1 K^{n_1} + \cdots + a_{t-1} K^{n_{t-1}} + y) \\
&= A(a_1 K^{n_1}) + a_1 a_2 K^{n_2} + A(a_2 K^{n_2}) + \cdots \\
&\quad + (a_1 + \cdots + a_{t-1}) a_t K^{n_t} + A(a_t K^{n_t}) \\
&= \sum_{i=1}^t A(a_i K^{n_i}) + \sum_{i=2}^t (a_1 + \cdots + a_{i-1}) a_i K^{n_i} \\
&= \sum_{i=1}^t \left[n_i \frac{a_i K^{n_i} (K-1)}{2} + K^{n_i} \frac{a_i (a_i - 1)}{2} + a_i \right] \\
&\quad + \sum_{i=2}^t \left(\sum_{j=1}^{i-1} a_j \right) a_i K^{n_i} \\
&= \frac{n_1 (K-1)}{2} \sum_{i=1}^t a_i K^{n_i} - \frac{K-1}{2} \sum_{i=1}^t (n_1 - n_i) a_i K^{n_i} \\
&\quad + \frac{1}{2} \sum_{i=1}^t a_i (a_i - 1) K^{n_i} + \sum_{i=1}^t a_i + \sum_{i=2}^t \left(\sum_{j=1}^{i-1} a_j \right) a_i K^{n_i} \\
&= t_1 - t_2 + t_3 + t_4 + t_5. \\
t_1 &= \frac{n_1 (K-1)}{2} x = \frac{(K-1)}{2} \left[\frac{\log x}{\log K} \right] x = \frac{K-1}{2} \frac{x \log x}{\log K} + O(x); \\
t_2 &= \frac{K-1}{2} \sum_{i=1}^t (n_1 - n_i) a_i K^{n_i} \leq \frac{(K-1)^2}{2} \sum_{i=1}^t (n_1 - n_i) K^{n_i} \\
&\leq \frac{(K-1)^2}{2} \sum_{j=0}^{n_1-1} (n_1 - j) K^j = \frac{(K-1)^2}{2} \sum_{j=0}^{n_1-1} \left(\sum_{i=0}^j K^i \right) \\
&= \frac{(K-1)^2}{2} \sum_{j=0}^{n_1-1} \frac{K^{j+1} - 1}{K-1} = \frac{K-1}{2} \left(\sum_{j=0}^{n_1-1} K^{j+1} - n_1 \right) \\
&< \frac{K-1}{2} \left(\frac{K^{n_1} - 1}{K-1} K \right) = \frac{K}{2} (K^{n_1} - 1) < \frac{K}{2} x, \text{ i.e., } t_2 = O(x); \\
t_3 &= \frac{1}{2} \sum_{i=1}^t a_i (a_i - 1) K^{n_i} < \frac{K-1}{2} \sum_{i=1}^t a_i K^{n_i} = \frac{K-1}{2} x, \\
&\text{i.e., } t_3 = O(x);
\end{aligned}$$

$$\begin{aligned}
 t_4 &= \sum_{i=1}^t a_i < x, \text{ i.e., } t_4 = O(x); \\
 t_5 &= \sum_{i=2}^t \left(\sum_{j=1}^{i-1} a_j \right) a_i K^{n_i} \leq (K-1)^2 \sum_{i=2}^t (i-1) K^{n_i} \\
 &= (K-1)^2 (K^{n_2} + 2K^{n_3} + \cdots + (t-1)K^{n_t}) \\
 &\leq (K-1)^2 [n_1 + (n_1-1)K + \cdots + 2K^{n_1-2} + K^{n_1-1}] \\
 &= (K-1)^2 \left[\sum_{j=0}^{n_1-1} \left(\sum_{i=0}^j K^i \right) \right] = (K-1)^2 \sum_{j=0}^{n_1-1} \frac{K^{j+1} - 1}{K - 1} \\
 &= (K-1) \left(\sum_{j=0}^{n_1-1} K^{j+1} - n_1 \right) \leq K(K-1) \frac{K^{n_1} - 1}{K - 1} \leq Kx, \\
 &\text{i.e., } t_5 = O(x).
 \end{aligned}$$

Therefore

$$A(x) = \frac{K-1}{2} \frac{x \log x}{\log K} + O(x).$$

At the special case of $x = (K-1)K$,

$$\frac{x \log x}{\log K} = \frac{(K-1)K^n \log K - (K-1)}{\log K}.$$

Hence

$$\begin{aligned}
 &\left(\frac{K-1}{2} \frac{x \log x}{\log K} - A(x) \right) / x = \left(\frac{(K-1)^2}{2} \frac{K^n [n \log K + \log(K-1)]}{\log K} \right. \\
 &\quad \left. - \frac{n(K-1)^2 K^n}{2} - \frac{(K-1)(K-2)K^n}{2} - (K-1) \right) / (K-1)K^n \\
 &= \frac{K-1}{2} \frac{\log(K-1)}{\log K} - \frac{K-2}{2} - \frac{1}{K^n}.
 \end{aligned}$$

As $n \rightarrow \infty$, the above formula's limit is larger than 0. Hence we have

$$\begin{aligned}
 &\limsup_{y \rightarrow \infty} \left(\frac{K-1}{2} \frac{y \log y}{\log K} \right) / y \\
 &\geq \lim_{x = (K-1)K^n, n \rightarrow \infty} \left[\frac{K-1}{2} \frac{x \log x}{\log K} - A(x) \right] / x > 0.
 \end{aligned}$$

At the same time in the special case of $x = K^n$, it is easy to prove

$$\lim_{x \rightarrow \infty} A(x) - \frac{K-1}{2} \frac{x \log x}{\log K} / x = 0.$$

This implies that our theorem is the best possible.

3. Proof of Theorem 2. Let m be a fixed integer, then the equation $\alpha(y) = m$ has infinite solutions. Letting $B_m(x)$ be the number of solutions not larger than x , we have Theorem 2.

THEOREM 2.

$$B_m(x) \sim \frac{1}{m!} \left(\frac{\log x}{\log K} \right)^m.$$

PROOF. First we set $m \leq K-1$ and $x = K^n$. If $y = a_0 + a_1K + \dots + a_{n-1}K^{n-1}$ and $\alpha(y) = m$, then in the expansion of $(x_0 + x_1 + x_2 + \dots + x_{n-1})^m$, there must exist a term which is $x_0^{a_0} \dots x_{n-1}^{a_{n-1}}$, or vice versa. This event shows that each term of the above formula's expansion has a one-to-one correspondence to y . Here y can satisfy such relations: $\alpha(y) = m$, $y \leq x$. But the above expansion has

$$\binom{n+m-1}{m}$$

terms, and

$$\binom{n+m-1}{m} \sim \frac{n^m}{m!} = \frac{1}{m!} \left(\frac{\log x}{\log K} \right)^m, \quad (n \rightarrow \infty)$$

so that our theorem is true with respect to this setting.

Secondly we set $m \leq K-1$ and $K^{n-1} < x < K^n$.

$$\begin{aligned} \left(\frac{n-1}{n} \right)^m \frac{B(K^{n-1})}{\frac{1}{m!} (n-1)^m} &= \frac{B(K^{n-1})}{\frac{1}{m!} n^m} < \frac{B_m(x)}{\frac{1}{m!} \left(\frac{\log x}{\log K} \right)^m} < \frac{B_m(K^n)}{\frac{1}{m!} (n-1)^m} \\ &= \frac{B_m(K^n)}{\frac{n^m}{m}} \left(\frac{n}{n-1} \right)^m. \end{aligned}$$

The above inequality's both ends approach 1 as $n \rightarrow \infty$. From this result we can derive

$$B_m(x) \sim \frac{1}{m!} \left(\frac{\log x}{\log K} \right)^m, \quad x \rightarrow \infty.$$

Now we investigate the case of $m \geq K$. With the same procedure

we set $x = K^n$. We classify the expansion of $(x_0 + x_1 + \cdots + x_{n-1})^m$ into A and B . A consists of such term $x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ which satisfies $a_i \leq K-1$ ($i=0, \cdots, n-1$); B consists of the other terms which do not belong to A . It is readily seen that B 's terms cannot be larger than n times the terms of expansion of the following formula:

$$(x_0 + x_1 + \cdots + x_{n-1})^{m-K}.$$

Hence B 's terms are not larger than

$$n \cdot \binom{n+m-K-1}{m-K}.$$

That is lower infinite of $n^m/m!$. In the beginning we speak of $K \geq 2$.

It is easy to see that y , which can satisfy the relationship $\alpha(y) = m$, $K^{n-1} < x < K^n$, is one-to-one corresponding to every term of B . Therefore,

$$B_n(K^n) \sim \frac{n^m}{m!}, \quad n \rightarrow \infty.$$

It is the same as the second situation, and we have

$$B_m(x) \sim \frac{1}{m!} \left(\frac{\log x}{\log K} \right)^m,$$

as $m \geq K$, $K^{n-1} < x < K^n$. Our theorem is fully proved.

REFERENCE

1. R. Bellman and H. Shapiro, *On a problem in additive number theory*, Ann. of Math. (2) 49 (1948), 333-340.

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