THE EXISTENCE OF FREE UNIONS IN CLASSES OF ABSTRACT ALGEBRAS

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Let κ be an ordinal number. A κ -ary operation on a set A is a mapping $O: A^\kappa \rightarrow A$ of all well ordered sequences (of type κ) of elements of A into A . Let $\langle \kappa_0, \dots, \kappa_\alpha, \dots \rangle_{\alpha < \beta}$ be a well ordered sequence of ordinal numbers. An *abstract algebra* of type $\langle \kappa_0, \dots, \kappa_\alpha, \dots \rangle_{\alpha < \beta}$ is a system $\langle A; O_0, \dots, O_\alpha, \dots \rangle_{\alpha < \beta}$, where A is a nonempty set and O_α is a κ_α -ary operation on A . The concepts of isomorphism, homomorphism, subalgebra, and direct union of abstract algebras are defined in the usual way (see [2]). Henceforth, we will not distinguish between the abstract algebra $\langle A; O_0, \dots, O_\alpha, \dots \rangle_{\alpha < \beta}$ and the set A of its elements. The concept of free union is defined relative to a class $\mathfrak{A}$ of algebras. It will always be assumed that the algebras in $\mathfrak{A}$ are of the same type. Explicit mention of this assumption will usually be omitted.

DEFINITION 1. Let $\mathfrak{A}$ be a class of abstract algebras. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a subset of $\mathfrak{A}$. A system $\langle A; h_\lambda \rangle_{\lambda \in \Lambda}$ consisting of an algebra $A \in \mathfrak{A}$ and a family of projections

$$h_\lambda: A \rightarrow A_\lambda$$

is called a free $\mathfrak{A}$ -union of the set $\{A_\lambda | \lambda \in \Lambda\}$ if the following extension property is satisfied:

(E) If $\{g_\lambda | \lambda \in \Lambda\}$ is a family of homomorphisms $g_\lambda: B \rightarrow A_\lambda$, where $B \in \mathfrak{A}$, then there is a unique homomorphism $g: B \rightarrow A$ such that $g_\lambda = h_\lambda \circ g$ for all λ .

This concept is dual to that of free $\mathfrak{A}$ -products studied by D. J. Christensen and R. S. Pierce in [3].

It is not clear that free $\mathfrak{A}$ -unions exist. However, if the free $\mathfrak{A}$ -union of a set of algebras does exist, then it is unique in the following sense: for any two free $\mathfrak{A}$ -unions

$$\langle A; h_\lambda \rangle_{\lambda \in \Lambda} \quad \text{and} \quad \langle A; \bar{h}_\lambda \rangle_{\lambda \in \Lambda}$$

of a set $\{A_\lambda | \lambda \in \Lambda\} \subseteq \mathfrak{A}$, there exist unique inverse isomorphisms

$$h: A \rightarrow A \quad \text{and} \quad \bar{h}: \bar{A} \rightarrow A$$

such that

Presented to the Society, August 31, 1962; received by the editors March 26, 1962.

¹ The author is indebted to Professor R. S. Pierce for his valuable suggestions during the preparation of this paper.

$$\bar{h}_\lambda = h_\lambda \circ \bar{h} \quad \text{and} \quad h_\lambda = \bar{h}_\lambda \circ h.$$

The existence and uniqueness of h and $\bar{h}$ come immediately from Definition 1. Since

$$\bar{h}_\lambda = \bar{h}_\lambda \circ h \circ \bar{h} \quad \text{and} \quad h_\lambda = h_\lambda \circ \bar{h} \circ h,$$

$h \circ \bar{h}$ and $\bar{h} \circ h$ are the identity mappings of $\bar{A}$ and A respectively by the uniqueness requirement in (E).

DEFINITION 2. Let $\mathfrak{A}$ be a class of abstract algebras. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a subset of $\mathfrak{A}$. Let $\langle A; h_\lambda \rangle_{\lambda \in \Lambda}$ be a system where $A \in \mathfrak{A}$ and h_λ is a homomorphism mapping A into A_λ . Define a relation $\sim$ on A as follows: for x and y in A , $x \sim y$ if and only if $h_\lambda(x) = h_\lambda(y)$ for all $\lambda \in \Lambda$. For $x \in A$, let $\hat{x} = \{y \in A | x \sim y\}$. Let $\hat{A} = \{\hat{x} | x \in A\}$. Let π be the mapping $A \rightarrow \hat{A}$ defined by $\pi(x) = \hat{x}$.

LEMMA 3. (i) $\sim$ is a congruence relation on A ; that is, an equivalence relation such that

$$x_0 \sim y_0, \dots, x_\mu \sim y_\mu, \dots, \quad (\mu < \kappa_\alpha)$$

implies $O_\alpha \langle x_0, \dots, x_\mu, \dots \rangle \sim O_\alpha \langle y_0, \dots, y_\mu, \dots \rangle$.

(ii) $\hat{A}$ is an abstract algebra of type $\langle \kappa_0, \dots, \kappa_\alpha, \dots \rangle$, with operations $\hat{O}_\alpha$ defined by

$$\hat{O}_\alpha \langle \hat{x}_0, \dots, \hat{x}_\mu, \dots \rangle = [O_\alpha \langle x_0, \dots, x_\mu, \dots \rangle]^\wedge.$$

Thus the mapping π is a homomorphism of A onto $\hat{A}$.

(iii) There is a unique family of homomorphisms $\tau_\lambda: \hat{A} \rightarrow A_\lambda$ such that $h_\lambda = \tau_\lambda \circ \pi$. Moreover, the τ_λ separate points. If $\hat{A} \in \mathfrak{A}$ and $\langle A; h_\lambda \rangle_{\lambda \in \Lambda}$ is a free $\mathfrak{A}$ -union of $\{A_\lambda | \lambda \in \Lambda\}$, so is $\langle \hat{A}, \tau_\lambda \rangle_{\lambda \in \Lambda}$.

PROOF. The proof of (i) and (ii) is routine.

(iii) Define $\tau_\lambda: \hat{A} \rightarrow A_\lambda$ by $\tau_\lambda(\hat{x}) = h_\lambda(x)$. τ_λ is clearly well defined and is a homomorphism since h_λ is. $\hat{x} \neq \hat{y}$ implies there exists λ such that $h_\lambda(x) \neq h_\lambda(y)$. Thus $\tau_\lambda(\hat{x}) \neq \tau_\lambda(\hat{y})$ and $\{\tau_\lambda | \lambda \in \Lambda\}$ separates points. Since h_λ is onto for all λ , τ_λ is onto for all λ since $h_\lambda = \tau_\lambda \circ \pi$. To show that $\langle \hat{A}, \tau_\lambda \rangle_{\lambda \in \Lambda}$ is the free $\mathfrak{A}$ -union of $\{A_\lambda | \lambda \in \Lambda\}$, suppose $g_\lambda: B \rightarrow A_\lambda$, where $B \in \mathfrak{A}$, is a set of homomorphisms. By (E) of Definition 1, there is a unique homomorphism $j: B \rightarrow A$ such that $g_\lambda = h_\lambda \circ j$. Define the mapping $g: B \rightarrow \hat{A}$ by $g = \pi \circ j$. Then

$$\tau_\lambda \circ g = \tau_\lambda \circ \pi \circ j = h_\lambda \circ j = g_\lambda.$$

The uniqueness of g follows from the fact that $\{\tau_\lambda | \lambda \in \Lambda\}$ separates points.

LEMMA 4. Let $\langle A; h_\lambda \rangle_{\lambda \in \Lambda}$ be a system having all the properties of

Definition 1 except possibly the uniqueness of the homomorphism g in (E). Assume that $\{h_\lambda | \lambda \in \Lambda\}$ separates points. Then g is unique.

Conversely, if any homomorphic image of an algebra of $\mathfrak{A}$ is in $\mathfrak{A}$ and $\langle A; h_\lambda \rangle_{\lambda \in \Lambda}$ is a free $\mathfrak{A}$ -union, then $\{h_\lambda | \lambda \in \Lambda\}$ separates points.

PROOF. Suppose g and $\bar{g}$ are homomorphisms of B such that $h_\lambda \circ g = h_\lambda \circ \bar{g}$ for all $\lambda \in \Lambda$. Let $b \in B$. Then $h_\lambda(g(b)) = h_\lambda(\bar{g}(b))$ for all λ implies $g(b) = \bar{g}(b)$ since $\{h_\lambda | \lambda \in \Lambda\}$ separates points.

To prove the converse, notice by Lemma 3, $\langle \hat{A}; \tau_\lambda \rangle_{\lambda \in \Lambda}$ is a free $\mathfrak{A}$ -union and $\{\tau_\lambda | \lambda \in \Lambda\}$ separates points. Hence there exist unique inverse isomorphisms $i: A \rightarrow \hat{A}$ and $\bar{i}: \hat{A} \rightarrow A$ such that $h_\lambda = \tau_\lambda \circ i$ and $\tau_\lambda = h_\lambda \circ \bar{i}$ for all λ . Thus $x \neq y$ implies $i(x) \neq i(y)$ which implies $\tau_\lambda(i(x)) \neq \tau_\lambda(i(y))$ for some λ . That is, $h_\lambda(x) \neq h_\lambda(y)$ for some λ .

COROLLARY 5. *Let $\mathfrak{A}$ be a class of abstract algebras with the property that any homomorphic image of an algebra of $\mathfrak{A}$ is in $\mathfrak{A}$. Let $\langle A; h_\lambda \rangle_{\lambda \in \Lambda}$ be the free union of $\{A_\lambda | \lambda \in \Lambda\}$. Assume $\{g_\lambda | \lambda \in \Lambda\}$ is a set of homomorphisms $g_\lambda: B \rightarrow A_\lambda$, where $B \in \mathfrak{A}$. Then the homomorphism $g: B \rightarrow A$ satisfying $g_\lambda = h_\lambda \circ g$ for all $\lambda \in \Lambda$ is one-to-one if and only if $\{g_\lambda | \lambda \in \Lambda\}$ separates points.*

PROOF. Suppose g is one-to-one and $x \neq y$ in B . Then $g(x) \neq g(y)$. By Lemma 4, there is a λ such that $h_\lambda(g(x)) \neq h_\lambda(g(y))$.

Conversely, suppose $\{g_\lambda | \lambda \in \Lambda\}$ separates points and $x \neq y$ in B . Then there is a λ such that $g_\lambda(x) \neq g_\lambda(y)$, or $h_\lambda(g(x)) \neq h_\lambda(g(y))$. This implies $g(x) \neq g(y)$.

Suppose the class $\mathfrak{A}$ is closed under the formation of direct unions. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a subset of $\mathfrak{A}$. Let A be the direct union of $\{A_\lambda | \lambda \in \Lambda\}$. Let $h_\lambda: A \rightarrow A_\lambda$ be the projection homomorphism. That is,

$$h_\lambda(\dots, x^\lambda, \dots) = x^\lambda, \quad x^\lambda \in A_\lambda.$$

Then it is easy to see that $\langle A; h_\lambda \rangle_{\lambda \in \Lambda}$ is the free $\mathfrak{A}$ -union of $\{A_\lambda | \lambda \in \Lambda\}$. Thus the existence of direct unions implies the existence of free $\mathfrak{A}$ -unions. However, free $\mathfrak{A}$ -unions may exist without the existence of direct unions in $\mathfrak{A}$.

We will conclude this paper with an existence theorem for free $\mathfrak{A}$ -unions. The following result for free $\mathfrak{A}$ -products was proved in [3]: Let $\mathfrak{A}$ be a class of abstract algebras with the following properties:

- (i) any algebra isomorphic to an algebra of $\mathfrak{A}$ is in $\mathfrak{A}$;
- (ii) any subalgebra of an algebra of $\mathfrak{A}$ is in $\mathfrak{A}$;
- (iii) any direct union of algebras of $\mathfrak{A}$ is in $\mathfrak{A}$.

Let $\{A_\lambda | \lambda \in \Lambda\}$ be a subset of $\mathfrak{A}$. Suppose that there is an algebra C in $\mathfrak{A}$ and a family of isomorphisms $j_\lambda: A_\lambda \rightarrow C$. Then the free $\mathfrak{A}$ -product of $\{A_\lambda | \lambda \in \Lambda\}$ exists.

It is clear from the proof of this theorem and the comment above that condition (iii) may be replaced by (iii)' any free $\mathfrak{A}$ -union of algebras of $\mathfrak{A}$ is in $\mathfrak{A}$. With this small change, we get an existence theorem for free $\mathfrak{A}$ -unions which dualizes this result. In the proof of this theorem, we need the following lemma.

LEMMA 6. Let $\mathfrak{A}$ be a class of abstract algebras. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a subset of $\mathfrak{A}$. Suppose $h_\lambda: C \rightarrow A_\lambda$, $C \in \mathfrak{A}$, is a family of mappings which separates points. Let $\beta = \prod_{\lambda \in \Lambda} |A_\lambda|$. Then $|C| \leq \beta$.

PROOF. Let A be the set product of $\{A_\lambda | \lambda \in \Lambda\}$. Then $|A| = \beta$. Define $\psi: C \rightarrow A$ by $\psi(x) = (\dots, h_\lambda(x), \dots)$. ψ is one-to-one: $x \neq y$ implies there is a λ such that $h_\lambda(x) \neq h_\lambda(y)$. Thus, $(\dots, h_\lambda(x), \dots) \neq (\dots, h_\lambda(y), \dots)$ or $\psi(x) \neq \psi(y)$. Hence $|C| \leq \beta$.

THEOREM 7. Let $\mathfrak{A}$ be a class of abstract algebras with the following properties:

- (i) any homomorphic image of an algebra of $\mathfrak{A}$ is in $\mathfrak{A}$;
- (ii) any free $\mathfrak{A}$ -product of algebras of $\mathfrak{A}$ is in $\mathfrak{A}$.

Let $\{A_\lambda | \lambda \in \Lambda\}$ be a subset of $\mathfrak{A}$. Suppose there is an algebra $C \in \mathfrak{A}$ and a family of homomorphisms $h_\lambda: C \rightarrow A_\lambda$ onto. Then the free $\mathfrak{A}$ -union of $\{A_\lambda | \lambda \in \Lambda\}$ exists.

PROOF. (1) Let $\beta = \prod_{\lambda \in \Lambda} |A_\lambda|$ as in Lemma 6. Choose a set M of cardinality β . Let

$$\mathfrak{B} = \{ \langle B_\delta; g_{\delta\lambda} \rangle_{\lambda \in \Lambda} \mid \delta \in \Delta \}$$

where $B_\delta \in \mathfrak{A}$, B_δ is a subset of M and for all $\lambda \in \Lambda$, $g_{\delta\lambda}$ is a homomorphism mapping B_δ into A_λ . Notice that $\mathfrak{B}$ is a set.

(2) Suppose $h_\lambda: B \rightarrow A_\lambda$, $B \in \mathfrak{A}$, is a set of homomorphisms. Then there is a $\delta \in \Delta$ and a homomorphism

$$g: B \rightarrow B_\delta$$

such that

$$h_\lambda = g_{\delta\lambda} \circ g \quad \text{for all } \lambda \in \Lambda.$$

By Lemma 3, $\hat{B}$ is an algebra and is the homomorphic image of B . So by (i), $\hat{B}$ is in $\mathfrak{A}$. Also by Lemma 3, there is a set of homomorphisms $g_\lambda: \hat{B} \rightarrow A_\lambda$ such that $h_\lambda = g_\lambda \circ \pi$ which separates points. Thus by Lemma 6, $|\hat{B}| \leq \beta$. Hence there is a one-to-one mapping $\gamma: \hat{B} \rightarrow M$. Let $N = \gamma(\hat{B})$. γ induces operations $O_0, \dots, O_\alpha, \dots, (\alpha < \beta)$, on N so that γ becomes an isomorphism. Thus N is an algebra in $\mathfrak{A}$ and is a subset of M . Furthermore, $g_\lambda \circ \gamma^{-1}$ is a set of homomorphisms from N into A_λ . Hence $\langle N; g_\lambda \circ \gamma^{-1} \rangle$ is in $\mathfrak{B}$. Let $N = B_\delta$, $g_{\delta\lambda} = g_\lambda \circ \gamma^{-1}$ and

$g = \gamma \circ \pi$ where π is the homomorphism of B onto $\hat{B}$. Then g is a homomorphism of B onto N satisfying $g_{\delta\lambda} \circ g = h_\lambda$ for all $\lambda \in \Lambda$.

Notice taking C in the last assumption of the theorem to be B , we have that $\mathfrak{B}$ is nonempty.

(3) Let $\langle D; i_\delta \rangle_{\delta \in \Delta}$ be the free $\mathfrak{A}$ -product of the algebras of $\mathfrak{B}$. Then there exist homomorphisms

$$\psi_\lambda: D \rightarrow A_\lambda$$

such that

$$g_{\delta\lambda} = \psi_\lambda \circ i_\delta.$$

By (2) and the last assumption of the theorem, there is a $\delta \in \Delta$ and a homomorphism $h: C \rightarrow B_\delta$ such that

$$h_\lambda = g_{\delta\lambda} \circ h \quad \text{for all } \lambda.$$

This implies $g_{\delta\lambda}$ is onto for all λ and hence ψ_λ is onto for all λ . Let $\tau_\lambda: \hat{D} \rightarrow A_\lambda$ be the mappings defined in Lemma 3. Then τ_λ is onto since ψ_λ is; and $\{\tau_\lambda | \lambda \in \Lambda\}$ separates points. It remains to show that $\langle \hat{D}; \tau_\lambda \rangle$ is the free $\mathfrak{A}$ -union of $\{A_\lambda | \lambda \in \Lambda\}$.

Suppose $h_\lambda: B \rightarrow A_\lambda$, $B \in \mathfrak{A}$, is a set of homomorphisms. By (2), there is a δ in Δ and a homomorphism

$$h: B \rightarrow B_\delta \text{ satisfying } h_\lambda = g_{\delta\lambda} \circ h.$$

The mapping $\eta = \pi \circ i_\delta \circ h$ is a homomorphism from B to $\hat{D}$ and

$$\tau_\lambda \circ \eta = \tau_\lambda \circ \pi \circ i_\delta \circ h = \psi_\lambda \circ i_\delta \circ h = g_{\delta\lambda} \circ h = h_\lambda.$$

Since $\{\tau_\lambda | \lambda \in \Lambda\}$ separates points, η is unique by Lemma 4.

COROLLARY 8. *If the class $\mathfrak{A}$ satisfies conditions (i) and (ii) of Theorem 7, and if in addition, every algebra in $\mathfrak{A}$ contains a one element subalgebra, then every subset of $\mathfrak{A}$ has a free $\mathfrak{A}$ -union.*

PROOF. We need only show that the last condition of Theorem 7 is satisfied. Let $\langle C; i_\lambda \rangle$ be the free $\mathfrak{A}$ -product of $\{A_\lambda | \lambda \in \Lambda\}$. For all $\mu \in \Lambda$, let $g_{\lambda\mu}$ be the identity mapping of A_μ if $\lambda = \mu$ and the unique homomorphism of A_λ onto the one element subalgebra of A_μ if $\lambda \neq \mu$. Then there exist homomorphisms $g_\mu: C \rightarrow A_\mu$, $\mu \in \Lambda$, such that

$$g_{\lambda\mu} = g_\mu \circ i_\lambda \quad \text{for all } \lambda \in \Lambda.$$

Since when $\mu = \lambda$, $g_{\lambda\mu}$ is onto, g_μ is onto for all μ .

COROLLARY 9. *Let $\mathfrak{A}$ be a class of algebras satisfying (i) and (ii) of Theorem 7. Suppose $\mathfrak{A}$ contains free $\mathfrak{A}$ -algebras of arbitrarily large generating sets. Then every subset of $\mathfrak{A}$ has a free $\mathfrak{A}$ -union.*

The free $\mathfrak{A}$ -algebra with γ generators is an algebra $B \in \mathfrak{A}$ containing a subset X of cardinality γ with the property that any mapping of X into an algebra $A \in \mathfrak{A}$ can be uniquely extended to a homomorphism of B into A .

PROOF OF COROLLARY 9. Let $\{A_\lambda | \lambda \in \Lambda\}$ be any subset of $\mathfrak{A}$. Let $\beta = \sup \{|A_\lambda| | \lambda \in \Lambda\}$. Let F be a free algebra in $\mathfrak{A}$ whose generating set, X , has cardinality greater than or equal to β . Since $|A_\lambda| \leq |X|$ for all λ , there exists a mapping of X onto A_λ which extends uniquely to a homomorphism of F onto A_λ . Thus the last condition of Theorem 7 is satisfied.

It is proved in [3] that if for the class $\mathfrak{A}$, (i) every algebra isomorphic to an algebra of $\mathfrak{A}$ is in $\mathfrak{A}$; (ii) every subalgebra of an algebra in $\mathfrak{A}$ is in $\mathfrak{A}$; (iii) every free $\mathfrak{A}$ -union of algebras in $\mathfrak{A}$ is in $\mathfrak{A}$; (iv) every algebra of $\mathfrak{A}$ contains a one element subalgebra; then every subset of $\mathfrak{A}$ has a free $\mathfrak{A}$ -product. This, together with 8, gives the following theorem.

THEOREM 10. *If the class $\mathfrak{A}$ satisfies:*

- (i) *any homomorphic image of an algebra in $\mathfrak{A}$ is in $\mathfrak{A}$;*
- (ii) *any subalgebra of an algebra of $\mathfrak{A}$ is in $\mathfrak{A}$;*
- (iii) *every algebra of $\mathfrak{A}$ contains a one element subalgebra; then every subset of $\mathfrak{A}$ has a free $\mathfrak{A}$ -product if and only if it has a free $\mathfrak{A}$ -union.*

REFERENCES

1. G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. **31** (1935), 434-454.
2. ———, *Lattice theory*, rev. ed., Amer. Math. Soc. Colloq. Publ. Vol. 25, Amer. Math. Soc., Providence, R. I., 1949.
3. D. J. Christensen and R. S. Pierce, *Free products of α -distributive boolean algebras*, Math. Scand. **7** (1959), 81-105.

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