ON THE QUOTIENT OF ENTIRE FUNCTIONS OF LOWER ORDER LESS THAN ONE

ANDERS HYLLENGREN

The origin of this note is a question in the theory of functions: "If $f_1(z)$ and $f_2(z)$ are two entire functions of lower order less than one and if $f_1(z)$ and $f_2(z)$ have the same zeros, is $f_1(z)/f_2(z)$ a constant?" This is one of 25 problems published in Bulletin of the American Mathematical Society, January, 1962, pp. 21–24.

The solution of this problem is that the quotient $f_1(z)/f_2(z)$ is not necessarily a constant. It is even possible to find such entire functions of lower order zero. To do this we introduce some definitions.

$$a_n = 2^{(4n)!}, \qquad b_n = 2^{(4n+2)!},$$

$$P_n(z) = \left(1 - \frac{z}{a_n}\right)^{a_n}\left(1 + \frac{z}{b_n}\right)^{b_n},$$

$$f_1(z) = \prod_{n=1}^{\infty} P_n(z), \qquad f_2(z) = e^{-z}f_1(z).$$

Now $f_1(z)$ and $f_2(z)$ are different entire functions with the same zeros. We denote

$$M_\nu(r) = \max_{|z|=r} |f_\nu(z)|, \qquad \nu = 1, 2.$$

We shall prove that the lower order of each of these functions is zero i.e.

$$\liminf_{r\to\infty} \frac{\log\log M_\nu(r)}{\log r} = 0, \qquad \nu = 1, 2.$$

We first estimate $\log\log M_1(r)$ for $r = 2^{(4m+3)!}$. Obviously, for $|z| = 2^{(4m+3)!}$ we have

$$|P_n(z)| = \left|1 - \frac{z}{a_n}\right|^{a_n}\left|1 + \frac{z}{b_n}\right|^{b_n} < |z|^{a_n+b_n}$$

i.e.,

$$\log|P_n(z)| < \log 2\cdot(4m+3)!(a_n + b_n)$$

which implies

Received by the editors March 16, 1962.

$$\log \left| P_1(z) \cdot \cdots \cdot P_m(z) \right| < \log 2 \cdot (4m+3)!(a_1 + b_1 + a_2 + b_2 + \cdots + a_m + b_m).$$

Roughly estimated

$$\log \left| P_1(z) \cdot \cdots \cdot P_m(z) \right| < (4m+3)!b_m = (4m+3)!2^{(4m+2)!}.$$

For $P_{m+n}(z)$, $n \geqq 1$, we use another estimate. The following simple inequalities are well known

$$\left| \log\left(1 - \frac{z}{a}\right)^a + z \right| \leqq \frac{|z|^2}{a} \qquad \text{for } |z| < \frac{a}{2},$$

$$\left| \log\left(1 + \frac{z}{b}\right)^b - z \right| \leqq \frac{|z|^2}{b} \qquad \text{for } |z| < \frac{b}{2}.$$

With $a = a_{m+n}$ and $b = b_{m+n}$ the estimate becomes

$$\begin{aligned}
\log \left| P_{m+n}(z) \right| &\leqq \left| \log P_{m+n}(z) \right| \\
&= \left| \log\left(1 - \frac{z}{a_{m+n}}\right)^{a_{m+n}} + \log\left(1 - \frac{z}{b_{m+n}}\right)^{b_{m+n}} \right| \\
&\leqq \left| \log\left(1 - \frac{z}{a_{m+n}}\right)^{a_{m+n}} + z \right| + \left| \log\left(1 + \frac{z}{b_{m+n}}\right)^{b_{m+n}} - z \right| \\
&\leqq \frac{|z|^2}{a_{m+n}} + \frac{|z|^2}{b_{m+n}} < \frac{2|z|^2}{a_{m+n}} = 2^{1+2(4m+3)!-(4m+4n)!} < 2^{-n}.
\end{aligned}$$

Thus the infinite product

$$P_{m+1}(z) \cdot P_{m+2}(z) \cdot \cdots \cdot P_{m+n}(z) \cdot \cdots$$

is estimated by

$$\log \left| P_{m+1}(z) \cdot P_{m+2}(z) \cdot \cdots \right| < \sum_{n=1}^{\infty} 2^{-n} = 1.$$

For

$$M_1(r) = \max_{|z|=r} \left| \prod_{n=1}^{\infty} P_n(z) \right|$$

we then get

$$\log M_1(r) < (4m+3)!2^{(4m+2)!} + 1 < 2^{2\cdot(4m+2)!}$$

$$\log \log M_1(r) < 2 \log 2 \cdot (4m+2)!$$

where $\log r = \log 2 \cdot (4m+3)!$. Thus

$$\frac{\log\log M_1(r)}{\log r} < \frac{2}{4m+3} < \frac{1}{m}\cdot$$

This estimate implies that the lower order of $f_1(z)$ is zero. We now consider $\log\log M_2(r)$ for $r = 2^{(4m+1)!}$ and in the same way as before we obtain

$$\begin{aligned}\log &\left| P_1(z)\cdot P_2(z)\cdot \;\cdots\; \cdot P_{m-1}(z)\right| \\ &< \log 2\cdot(4m+1)!(a_1+b_1+\cdots+a_{m-1}+b_{m-1}) \\ &< (4m+1)!\, b_{m-1} = (4m+1)!2^{(4m-2)!}.\end{aligned}$$

Then we consider $P_m(z)\cdot e^{-z}$. We have

$$\begin{aligned}\log &\left| P_m(z)e^{-z}\right| \\ &\leqq \log\left|1-\frac{z}{a_m}\right|^{a_m} + \left|\log\left(1+\frac{z}{b_m}\right)^{b_m} - z\right| \\ &< \log|z|\, a_m + \frac{|z|^2}{b_m} = \log 2(4m+1)!2^{(4m)!} + 2^{2\cdot(4m+1)!-(4m+2)!} \\ &< (4m+1)!2^{(4m)!}+1.\end{aligned}$$

For $P_{m+n}(z)$ we obtain as before

$$\log\left|P_{m+n}(z)\right| < 2^{-n}, \qquad n \geqq 1.$$

Hence

$$\log\left|P_{m+1}(z)\cdot P_{m+2}(z)\cdot\;\cdots\right| < 1.$$

For

$$M_2(r) = \max_{|z|=r}\left|P_1(z)\cdot\;\cdots\;\cdot P_{m-1}(z)\cdot P_m(z)\cdot e^{-z}\cdot P_{m+1}(z)\cdot\;\cdots\right|$$

we obtain

$$\log M_2(r) < (4m+1)!2^{(4m-2)!} + (4m+1)!2^{(4m)!} + 1 + 1 < 2^{2\cdot(4m)!}.$$

Now $\log\log M_2(r) < 2\log 2\cdot(4m)!$ and $\log r = \log 2\cdot(4m+1)!$. Thus

$$\frac{\log\log M_2(r)}{\log r} < \frac{2}{4m+1} < \frac{1}{m}$$

which implies that the lower order of $f_2(z)$ is zero. The proof is now complete.

KUNGL. TEKNISKA HÖGSKOLAN, STOCKHOLM