SOME COMPLETE CYCLES ON THE n -CUBE

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1. Let Q_n^* be the n -dimensional hypercube (or n -cube) whose vertices are the 2^n vectors of dimension n with components 0's and 1's. Let Q_n be the graph that consists of the edges and vertices of Q_n^* . A complete cycle (or Hamilton circuit) on Q_n is a cyclic path on the graph Q_n that passes through each vertex once and only once. Let C be such a cycle and let $P_1, P_2, \dots, P_{2^n} = P_0$ be the consecutive vertices of C . Any two consecutive vertices P_i, P_{i+1} differ in exactly one component. Let A_i denote the index of that component. Then A_i is called the i th change number of C . The subscripts i in the P_i and the A_i are taken modulo 2^n . Clearly C is determined by P_1 and the sequence of 2^n change numbers

$$(1) \quad A_1, A_2, \dots, A_{2^n} = A_0.$$

Gilbert [1] has given a necessary and sufficient condition on (1) in order that it be the sequence of change numbers of a complete cycle on Q_n .

We will say that C traverses an r -cube R if the 2^r vertices of R occur as consecutive vertices of C .

The complete cycles on Q_3 are easy to determine. They are all equivalent under the group of symmetries of Q_3 , and each one of them traverses four squares. For $n=4$ the complete cycles have been enumerated [1]. They are of 9 distinct types, all of which traverse squares. However, one of them does not traverse any 3-cubes. For $n \geq 5$ I will show the existence of complete cycles that do not traverse any r -cubes, $2 \leq r \leq n-1$. The proof is by induction.

2. **The case $n=5$.** In this case we assert that

$$(2) \quad \begin{array}{l} 1, 2, 3, 5, 2, 3, 4, 5, 3, 4, 1, 5, 4, 1, 2, 5, \\ 1, 2, 3, 5, 2, 3, 4, 5, 3, 4, 1, 5, 4, 1, 2, 5 \end{array}$$

is the sequence of change numbers of a complete cycle on Q_5 that does not traverse any 2, 3, or 4-cubes. Let A_i denote the i th number in the sequence (2). We first show that this sequence is the sequence of change numbers of a complete cycle on Q_5 . We may take $P_1 = (00000)$. Then $P_2 = (10000)$, $P_3 = (11000)$, $P_4 = (11100)$, $P_5 = (11101)$, $P_6 = (10101)$, $\dots$. In Table I the fact that $P_i = (fghjk)$ is indicated by an

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i in the (fg) row and the (hjk) column. Inspecting Table I we see that we have indeed a complete cycle.

TABLE I
A COMPLETE CYCLE ON THE 5-CUBE

	(000)	(001)	(010)	(011)	(100)	(101)	(110)	(111)
(00)	1	32	20	21	12	13	19	14
(01)	26	31	27	22	25	24	18	23
(10)	2	7	9	8	11	6	10	15
(11)	3	30	28	29	4	5	17	16

Now $P_j, P_{j+1}, P_{j+2}, P_{j+3}$ are the vertices of a square if and only if $A_j = A_{j+2}$. (Here the subscripts are taken modulo 32.) A brief inspection of (2) shows that $A_j \neq A_{j+2}$ for all j . Hence our complete cycle traverses no squares.

In the sequence (2) the longest gap between successive identical integers is 10. Hence any set of 10 consecutive integers in (2) contains all five integers 1, 2, 3, 4, 5. The same is clearly true for any set of 15 consecutive integers in (2). This implies that our complete cycle traverses no 4-cubes.

To complete the discussion we need to show that our complete cycle traverses no 3-cubes.

LEMMA. *It is impossible to traverse a 3-cube without traversing any squares.*

PROOF. Suppose $P_j, P_{j+1}, \dots, P_{j+6}$ all lie on a 3-cube. Let $A_j, \dots, A_{j+5}$ be the corresponding change numbers. No four of these change numbers can be distinct. Suppose no squares are traversed. Then A_j, A_{j+1}, A_{j+2} are distinct and $A_{j+3} \neq A_{j+1}, A_{j+2}$. Hence $A_{j+3} = A_j$. Similarly $A_{j+4} = A_{j+1}$, $A_{j+5} = A_{j+2}$. It follows that $P_{j+6} = P_j$, a contradiction. This proves the lemma.

This lemma completes the proof of the fact that the complete cycle with change numbers (2) does not traverse any 2, 3, or 4-cubes.

3. The induction from n to $n+1$. Let $n \geq 5$, and let C_n be a complete cycle on the n -cube Q_n . Let $P_1 = (000 \dots 0)$, $P_2, \dots, P_{2^n} = P_0$ be the sequence of vertices of C_n , and let $A_1, A_2, \dots, A_{2^n} = A_0$ be the corresponding sequence of change numbers. We suppose

- (i) C_n traverses no r -cubes, $2 \leq r \leq n-1$.
- (ii) $P_1 P_2 P_{11} P_{12}$ are the consecutive vertices of a square S .
- (iii) $A_0 = 4$ or 5 ; $A_2 \neq 4, 5$; $A_{10} = 4$; $A_{12} = 5$.

We note that these conditions are satisfied by the complete cycle

C_5 on Q_5 constructed in §2.

Set $A = A_1$. Then by (ii) we have $A = A_1 = A_{11}$. Moreover, $P_2 P_{11}$ is an edge of Q_n . Hence P_2 and P_{11} differ in only one component. Let d denote the index of that component. Thus the sequence of change numbers of the square S is A, d, A, d . Moreover, since P_{10}, P_{13} are not on S we have $A \neq A_{10}, A_{12}$. In particular $A \neq 4, 5$ by (iii). Similarly, $d \neq 4, 5$. Let σ be any fixed permutation of $1, 2, \dots, n$ such that

$$(3) \quad \sigma A = d, \quad \sigma d = A, \quad \sigma 4 = 5, \quad \sigma 5 = 4.$$

For any vertex $P = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ of Q_n put $\tau = \sigma^{-1}$

$$\sigma P = (\epsilon_{\tau 1}, \epsilon_{\tau 2}, \dots, \epsilon_{\tau n}),$$

$$\lambda P = (\epsilon_1, \epsilon_2, \dots, \epsilon_n, 0),$$

$$\mu P = (\epsilon_1, \epsilon_2, \dots, \epsilon_n, 1).$$

In particular, $\sigma P_1 = P_1$, $\sigma P_2 = P_{12}$, $\sigma P_{11} = P_{11}$, and $\sigma P_{12} = P_2$.

We assert that

$$(4) \quad \begin{aligned} & n+1, A_2, A_3, \dots, A_{10}, n+1, \sigma A_{10}, \sigma A_9, \dots, \sigma A_2, \\ & n+1, A_{12}, A_{13}, \dots, A_{2^n}, n+1, \sigma A_{2^n}, \dots, \sigma A_{12} \end{aligned}$$

is the sequence of change numbers of a complete cycle on the $(n+1)$ -cube and that this complete cycle satisfies conditions (i), (ii), and (iii). Let b_i denote the i th number in the sequence (4). Thus

$$(5) \quad b_1 = b_{11} = b_{21} = b_{2^n+11} = n+1,$$

$$(6) \quad b_i = A_i, \quad 2 \leq i \leq 10,$$

$$(7) \quad b_i = \sigma A_{22-i}, \quad 12 \leq i \leq 20,$$

$$(8) \quad b_i = A_{i-10}, \quad 22 \leq i \leq 2^n + 10,$$

$$(9) \quad b_i = \sigma A_{2^{n+1}+12-i}, \quad 2^n + 12 \leq i \leq 2^{n+1}.$$

Let $B_1, B_2, \dots$ be the sequence of vertices of a path on Q_{n+1} with sequence of change numbers (4). We may choose B_1 arbitrarily. For convenience we take

$$B_1 = \mu P_2 = \mu \sigma P_{12}.$$

Since $b_1 = n+1$ we have $B_2 = \lambda P_2$. Using (6) this gives us

$$(10) \quad B_i = \lambda P_i, \quad 2 \leq i \leq 11.$$

Since $B_{11} = \lambda P_{11}$ and $b_{11} = n+1$, we have $B_{12} = \mu P_{11} = \mu \sigma P_{11}$. Now σP_i and σP_{i+1} differ only in the σA_i th component. Hence, using (7), we obtain $B_i = \mu \sigma P_{23-i}$, $12 \leq i \leq 21$. Thus $B_{21} = \mu \sigma P_2 = \mu P_{12}$ so that

$B_{22} = \lambda P_{12}$. From (8) we get

$$(11) \quad B_i = \lambda P_{i-10}, \quad 22 \leq i \leq 2^n + 11,$$

in particular, $B_{2^{n+11}} = \lambda P_1 = \lambda \sigma P_1$, so that $B_{2^{n+12}} = \mu \sigma P_1 = \mu \sigma P_{2^{n+1}}$. Hence $B_i = \mu \sigma P_{2^{n+1}+13-i}$, $2^n + 12 \leq i \leq 2^{n+1} + 1$, by (9). In particular $B_{2^{n+1}+1} = \mu \sigma P_{12} = B_1$. We see from these relations that the path $B_1, B_2, \dots$ passes through every vertex of Q_{n+1} . Hence we have a complete cycle. We denote this complete cycle by C_{n+1} .

We note that $b_0 = b_{2^{n+1}} = \sigma A_{12} = 4$; $b_2 = A_2 \neq 4, 5$; $b_{10} = A_{10} = 4$; and $b_{12} = \sigma A_{10} = 5$. Thus (iii) holds for C_{n+1} . Moreover, $B_1 = \mu P_2$, $B_2 = \lambda P_2$, $B_{11} = \lambda P_{11}$, $B_{12} = \mu P_{11}$. Since $P_2 P_{11}$ is an edge of Q_n it follows that B_1, B_2, B_{11}, B_{12} are the consecutive vertices of a square. Thus (ii) holds for C_{n+1} . To complete the discussion we must show that (i) holds, i.e., that C_{n+1} traverses no r -cube, $2 \leq r \leq n$.

Suppose T is an r -cube traversed by C_{n+1} , $2 \leq r \leq n$. It follows easily from the induction hypothesis and the sequence (4) that T is not contained in the n -cube λQ_n or the n -cube μQ_n . Hence $T \cap \lambda Q_n$ is an $(r-1)$ -cube. Since C_{n+1} traverses T it follows from (10) and (11) that C_n traverses the $(r-1)$ -cube $\lambda^{-1}(T \cap \lambda Q_n)$. Hence $r-1=1$, $r=2$, and T is a square. Thus for some j ; $B_j, B_{j+1}, B_{j+2}, B_{j+3}$ are the vertices of the square T . Hence $b_j = b_{j+2}$. By construction $b_j = b_{j+2} = n+1$ is impossible. Since T does not lie in either λQ_n or μQ_n one of the change numbers of T is $n+1$. Hence $b_{j+1} = n+1$. Therefore $j=0, 10, 20$, or 2^n+10 . Now

$$\begin{aligned} b_0 &= \sigma A_{12} = 4 \neq A_2 = b_2, \\ b_{10} &= 4 \neq 5 = b_{12}, \\ b_{20} &= \sigma A_2 \neq \sigma 4 = 5 = A_{12} = b_{22}, \end{aligned}$$

and finally, since $A_{2^n} = A_0 = 4$ or 5 we have

$$b_{2^n+10} = A_{2^n} \neq \sigma A_{2^n} = b_{2^n+12}.$$

It follows that $b_j \neq b_{j+2}$, a contradiction. Therefore, C_{n+1} traverses no r -cubes, $2 \leq r \leq n$. Thus we have proved:

THEOREM. *For any $n \geq 5$ there exists a complete cycle on the n -cube that does not traverse any r -cube, $2 \leq r \leq n-1$.*

REFERENCE

1. E. N. Gilbert, *Gray codes and paths on the n -cube*, Bell System Tech. J. **37** (1958), 815-826.

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