potents in each semigroup was a finite discrete set. It might be of interest to know if there exists a semigroup $S = ESE$ which is compact connected, has a zero, is not acyclic and such that the set of idempotents is connected.

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CONFORMAL VECTOR FIELDS IN COMPACT RIEMANNIAN MANIFOLDS

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1. **Introduction.** Let V^n be a compact Riemannian manifold of dimension n and of class C^3 . Let $g_{ij}(x)$ of class C^2 be the coefficients of the fundamental metric which is assumed to be positive definite. Let Γ_{ij}^h be the Christoffel symbol, R_{ijhk} the curvature tensor and R_{ij} the Ricci tensor.

Let ϕ be an arbitrary scalar invariant, ξ^i an arbitrary vector field and $\xi_{i_1 i_2} \dots i_p$ an arbitrary anti-symmetric tensor field of order p , all of class C^2 in V^n . We shall make use of the following results obtained by S. Bochner and K. Yano [1, pp. 31, 51, 69]:

$$(1.1) \quad (\Delta\phi \geq 0 \text{ everywhere in } V^n) \Rightarrow (\phi = \text{constant everywhere in } V^n).$$

$$(1.2) \quad \int_{V^n} \xi^i_{,i} dv = 0.$$

$$(1.3) \quad \int_{V^n} (R_{ij} \xi^i \xi^j + \xi^i_{,j} \xi^j_{,i} - \xi^i_{,i} \xi^j_{,j}) dv = 0.$$

$$(1.4) \quad \int_{V^n} (F\{\xi_{i_1 i_2} \dots i_p\} + \xi^{i i_2 \dots i_p, i} \xi_{j i_2 \dots i_p, i} - \xi^{i i_2 \dots i_p, i} \xi^j_{i_2 \dots i_p, j}) dv = 0$$

where

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$$F\{\xi_{i_1 i_2 \dots i_p}\} = R_{ij} \xi^{i i_2 \dots i_p} \xi^j_{i_2 \dots i_p} + \frac{p-1}{2} R_{ijkl} \xi^{ij i_3 \dots i_p} \xi^{kl}_{i_3 \dots i_p}$$

and V^n in equations (1.2) to (1.4) is assumed to be orientable.

If a vector field ξ^i defines a one-parameter continuous group of infinitesimal conformal transformations in V^n , the intrinsic derivative of $\xi_i dx^i/ds$ along any geodesic $x^i(s)$ depends only on the point and not on the direction of the geodesic passing through the point; that is

$$(1.5) \quad \xi_{i,j} \frac{dx^i}{ds} \frac{dx^j}{ds} = \frac{1}{n} \xi_{i,i}.$$

Let ξ^i be an arbitrary vector field which does not necessarily define a one-parameter continuous group of infinitesimal conformal transformations. Then, instead of (1.5), we have in general

$$\xi_{i,j} \frac{dx^i}{ds} \frac{dx^j}{ds} = f\left(x^i, \xi^i, \frac{dx^i}{ds}\right).$$

In the following, such ξ^i will be associated with an arbitrary but fixed scalar field ϕ in a certain way to give a generalization of some well-known vector fields.

2. Definition. Let λ^i be an arbitrary unit vector field. Consider M defined by

$$(2.1) \quad M = \xi_{i,j} \lambda^i \lambda^j - \phi.$$

When ξ^i defines a one-parameter continuous group of infinitesimal conformal transformations and $\phi = \xi^i_{,i}/n$, we have, by (1.5), $M = 0$. Hence it seems appropriate to call M the ϕ -conformal measure of ξ^i with respect to λ^i .

Let $\lambda_\alpha|{}^i$, $\alpha = 1, 2, \dots, n$, be n mutually orthogonal unit vector fields in V^n and M_α the ϕ -conformal measures of ξ^i with respect to them, that is

$$M_\alpha = \xi_{i,j} \lambda_\alpha|{}^i \lambda_\alpha|{}^j - \phi.$$

The mean of M_α is equal to

$$(2.2) \quad \frac{1}{n} \sum_{\alpha=1}^n M_\alpha = \frac{1}{n} g^{ij} \xi_{i,j} - \phi$$

which is evidently independent of the choice of the orthogonal ennuple $\lambda_\alpha|{}^i$. A vector field ξ^i for which the mean (2.2) vanishes is

called a ϕ -conformal vector field in V^n or simply a conformal vector field in V^n . To each ϕ there corresponds one conformal vector field. The definition leads immediately to the following necessary and sufficient condition

$$(2.3) \quad \xi^i_{,i} = n\phi.$$

Since a harmonic vector field and a Killing vector field satisfy (2.3) when $\phi = 0$, a conformal vector field may be considered as a generalization of them in this sense.

We intend to investigate properties of conformal vector fields in a compact Riemannian manifold V^n , particularly the global non-existence of these fields in V^n . A generalization of our concept to tensor fields is given at the end of the paper.

3. Properties. By definition of the Laplacean (denoted by Δ) of a scalar field, we obtain for a conformal vector field ξ^i

$$(3.1) \quad \Delta \xi^i_{,i} = g^{ik} \xi^i_{,ijk} = n\Delta\phi.$$

The tendency of a vector v^i in a unit direction a^i is defined as the projection of the vector $a^k v^i_{,k}$ in the direction of a^i . It is well known that the divergence of a vector in V^n is the sum of tendencies of the vector for n mutually orthogonal directions in V^n and that of a unit vector in V^2 is numerically equal to the geodesic curvature of a curve normal to the vector [2, p. 422; 3, p. 201]. Hence from (1.1) and (1.2) we have the following two theorems.

THEOREM 3.1. *If ϕ satisfies $\Delta\phi \geq 0$ everywhere in a compact Riemannian manifold V^n , then the sum of tendencies of the ϕ -conformal vector field for n mutually orthogonal directions is constant throughout the manifold. If ϕ satisfies $\Delta\phi \geq 0$ everywhere in a compact Riemannian manifold V^2 , then the geodesic curvature of the orthogonal trajectories of the curves of the ϕ -conformal unit vector field is constant throughout the manifold.*

THEOREM 3.2. *In a compact orientable Riemannian manifold V^n , there exists no ϕ -conformal vector field with $\phi > 0$ or $\phi < 0$ everywhere in the manifold and therefore the divergence of a ϕ -conformal vector field is a constant everywhere in V^n if and only if the constant is zero. In a compact orientable Riemannian manifold V^2 , the orthogonal trajectories of the curves of a ϕ -conformal unit vector field form a family of geodesics in the manifold if and only if the divergence of the vector field is constant throughout.*

By (2.3) we may write (1.3) as

$$\int_{V^n} (\xi^i_{,j} \xi^j_{,i} + R_{ij} \xi^i \xi^j) dv = \int_{V^n} n^2 \phi^2 dv$$

and

$$\int_{V^n} (\xi^i_{,j} \xi^j_{,i} - n^2 \phi^2) dv = - \int_{V^n} R_{ij} \xi^i \xi^j dv$$

which lead to the following two theorems:

THEOREM 3.3. *In a compact orientable Riemannian manifold V^n there exists no ϕ -conformal vector field ξ^i which satisfies*

$$\int_{V^n} (R_{ij} \xi^i \xi^j + \xi^i_{,j} \xi^j_{,i}) dv \leq 0$$

unless we have $\phi = 0$ and then automatically the equality sign holds.

THEOREM 3.4. *In a compact orientable Riemannian manifold with negative (positive) definite Ricci curvature throughout, there exists no ϕ -conformal vector field other than zero vector field which satisfies*

$$\int_{V^n} (\xi^i_{,j} \xi^j_{,i} - n^2 \phi^2) dv \leq 0 \quad (\geq 0).$$

The above theorem includes as special cases some results about global nonexistence of harmonic vector field and Killing vector field obtained by S. Bochner [1, pp. 37, 39].

4. Generalization. An anti-symmetric tensor field $\xi_{i_1 i_2 \dots i_p}$ of order p is a conformal Killing tensor field if and only if

$$(4.1) \quad \xi_{i_1 i_2 \dots i_p, j} \frac{dx^i}{ds} \frac{dx^j}{ds} = \phi_{i_1 i_2 \dots i_p}$$

where

$$\phi_{i_1 i_2 \dots i_p} = \frac{1}{n} g^{ij} \xi_{i_1 i_2 \dots i_p, j}$$

is an anti-symmetric tensor of order $p-1$ [1, p. 73].

Denote by $\lambda_\alpha |^i$ n mutually orthogonal unit vector fields, by $\phi_{i_1 i_2 \dots i_p}$ an arbitrary but fixed anti-symmetric tensor field of order $p-1$, and by $M_\alpha |_{i_2 \dots i_p}$ the following tensor fields

$$(4.2) \quad M_\alpha |_{i_2 \dots i_p} = \xi_{i_1 i_2 \dots i_p, j} \lambda_\alpha |^i \lambda_\alpha |^j - \phi_{i_2 \dots i_p}.$$

The mean of (4.2) is equal to

$$(4.3) \quad \frac{1}{n} \sum_{\alpha=1}^n M_{\alpha} |_{i_2 \dots i_p} = \frac{1}{n} \xi_{i_2 \dots i_p, j} g^{ij} - \phi_{i_2 \dots i_p}$$

which is independent of the choice of the orthogonal ennuple $\lambda_{\alpha} |^i$. An anti-symmetric tensor field $\xi_{i_1 \dots i_p}$ for which (4.3) vanishes is called a ϕ -conformal tensor field in V^n . Thus $\xi_{i_1 \dots i_p}$ is a ϕ -conformal tensor field in V^n if and only if

$$(4.4) \quad g^{ij} \xi_{i_2 \dots i_p, j} = n \phi_{i_2 \dots i_p}.$$

Obviously, a ϕ -conformal tensor field includes a conformal Killing tensor field as a special case.

Substituting (4.4) into (1.4) gives

THEOREM 4.1. *In a compact orientable Riemannian manifold there exists no ϕ -conformal tensor field of order p which satisfies*

$$F\{\xi_{i_1 \dots i_p}\} + \xi^{i i_2 \dots i_p, j} \xi_{j i_2 \dots i_p, i} \leq 0$$

unless we have $\phi_{i_2 \dots i_p} = 0$ and then automatically the equality sign holds.

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