COLUMN SEQUENCES IN HAUSDORFF MATRICES

J. D. BUCKHOLTZ

Corresponding to each sequence d of complex numbers, the Hausdorff matrix $H=H(d)$ is given by

$$H_{nk} = \begin{cases} 0 & \text{if } n < k, \\ \binom{n}{k} \Delta^{n-k} d_k & \text{if } n \geqq k, \end{cases} \qquad n, k = 0, 1, \cdots.$$

For convenience, we shall denote the kth column sequence by $h^{(k)}$, i.e., $h_n^{(k)} = H_{nk}$, $n=0, 1, \cdots$. For each $k \geqq 1$, C^k will denote the kth power of the Cesàro matrix $(C, 1)$. We shall make use of the fact that, regarded as summability methods, C^k and (C, k) are equivalent [1, p. 103]. If the C^k transform (and consequently the (C, k) transform) of a sequence s has limit x, we shall abbreviate this by $s_n \to x$ (C, k).

For Hausdorff matrices which satisfy the condition

$$\sum_{k=0}^{n} \left| H_{nk} \right| \leqq M \qquad (M \text{ independent of } n),$$

it is well known [1, p. 255] that $h^{(0)}$ converges and that every other column sequence converges to zero. The purpose of this note is to obtain a weaker form of this result for all Hausdorff matrices for which $h^{(0)}$ converges.

THEOREM. *If H is a Hausdorff matrix and $h^{(0)}$ converges, then $h_n^{(k)} \to 0$ (C, k) for every positive integer k.*

PROOF. The proof depends mainly on the sequence identity

$$(1) \qquad Ch^{(k)} = Ch^{(k-1)} - \frac{1}{k} h^{(k-1)},$$

where $Ch^{(k)}$ denotes the C transform of the sequence $h^{(k)}$. Noting that, for $n \geqq k$, the nth term of $Ch^{(k)}$ is

$$\frac{1}{n+1} \sum_{p=k}^{n} \binom{p}{k} \Delta^{p-k} d_k,$$

a verification of (1) follows from the identities

$$\Delta^{p-k} d_k = \Delta^{p-k} d_{k-1} - \Delta^{p-k+1} d_{k-1} \quad \text{and} \quad \binom{p}{k} - \binom{p-1}{k} = \binom{p-1}{k-1}.$$

Presented to the Society, July 5, 1962; received by the editors July 22, 1962.

If in (1), $k=1$, convergence of $h^{(0)}$ immediately implies $h_n^{(1)} \to 0$ $(C, 1)$. Suppose now that $k-1$ is a positive integer for which $h_n^{(k-1)} \to 0$ $(C, k-1)$. Applying the C^{k-1} matrix to both sides of (1), we have

$$C^k h^{(k)} = C^k h^{(k-1)} - \frac{1}{k} C^{k-1} h^{(k-1)}.$$

Since $C^{k-1}h^{(k-1)}$ has limit zero, so does $C^k h^{(k-1)}$, and therefore $C^k h^{(k)}$ has limit zero. This completes the proof.

COROLLARY. *If H is a Hausdorff matrix and $h^{(0)}$ converges, then for every positive integer k for which $h^{(k)}$ converges, $h^{(k)}$ has limit zero.*

PROOF. (C, k) summability is regular.

REFERENCE

1. G. H. Hardy, *Divergent series*, Clarendon, Oxford, 1949.

UNIVERSITY OF NORTH CAROLINA