THE ZEROS OF INFRAPOLYNOMIALS WITH PRESCRIBED VALUES AT GIVEN POINTS[1]

J. L. WALSH AND O. SHISHA

1. Hitherto there have been considered [e.g., 1956, 1961][2] infrapolynomials with some prescribed coefficients, that is to say, with prescribed values of certain derivatives at the point $z=0$. We now generalize this concept by prescribing values of the polynomial and of certain of its derivatives at given points $z_1, z_2, \cdots, z_k$, and study the geometric location of its zeros in the complex plane. Thus the present results are, broadly speaking, generalizations of the preceding ones.

2. Let $z_1, z_2, \cdots, z_k$ be (distinct) points of the (open) complex plane, and for each j $(=1, 2, \cdots, k)$, let there be given complex values $w_j^{(0)}, w_j^{(1)}, \cdots, w_j^{(m_j)}$. Let Λ be the set of all polynomials $A(z)$ satisfying

$$A^{(\nu)}(z_j) = w_j^{(\nu)} \qquad \nu = 0, 1, \cdots, m_j, \;\; j = 1, 2, \cdots, k.$$

We make the following

DEFINITION. *Let n be a positive integer, and let S be a pointset in the (open) complex plane. An (n, Λ, S) infrapolynomial is an element $A(z)$ of Λ of degree[3] $\leqq n$, having the property: there does not exist a polynomial $B(z)$ belonging to Λ and of degree $\leqq n$ such that*

$$B(z) \not\equiv A(z),$$

$$|B(z)| < |A(z)| \quad \textit{whenever } z \in S \textit{ and } A(z) \neq 0,$$

$$B(z) = 0 \quad \textit{whenever } z \in S \textit{ and } A(z) = 0\,.$$

3. We set $T=\{z_1, z_2, \cdots, z_k\}$, and denote by $P(z)$ the unique element of Λ of degree $\leqq -1+M$, where $M=\sum_{j=1}^{k}(m_j+1)$ [cf. 1935, Chapter III, Theorem 2]. Let n be a positive integer, and S a finite (nonempty) pointset in the (open) complex plane whose (distinct) elements we denote by $\zeta_1, \zeta_2, \cdots, \zeta_N$; we assume that S and T are disjoint.

Presented to the Society, April 30, 1962 under the title *Infrapolynomials with prescribed derivatives at given points*; received by the editors June 27, 1962.

[1] This research was supported (in part) by the U. S. Air Force through the Air Force Office of Scientific Research. Abstract, Notices Amer. Math. Soc. 9 (1962), 315.

[2] Dates in square brackets refer to the bibliography.

[3] The degree of a polynomial is understood to be its exact degree. The polynomial 0 is assigned -1 as its degree.

4. THEOREM 1.[4] *Let* $L(z) \equiv L_{N-1}z^{N-1} + \cdots + L_0$ *be Lagrange's interpolation polynomial to* $P(z)/Q(z)$ *on* S, *where* $Q(z) \equiv \prod_{j=1}^{k} (z-z_j)^{m_j+1}$, *and let* $A_0(z) \equiv P(z) - L(z)Q(z)$.

(a) *If* $N \leqq n - M$, *then there exists no* (n, Λ, S) *infrapolynomial.*

(b) *If* $N = n - M + 1$, *then* $A_0(z)$ *is the unique* (n, Λ, S) *infrapolynomial.*

(c) *If* $N = n - M + 2$, *then a polynomial* $A(z)$ *is an* (n, Λ, S) *infrapolynomial if and only if it is of the form*

$$A_0(z) + L_{N-1}Q(z) \sum_{\nu=1}^{N} \lambda_\nu \frac{g(z)}{z - \zeta_\nu} \tag{1}$$

where $\lambda_\nu \geqq 0$, $\sum_{\nu=1}^{N} \lambda_\nu = 1$. *Here* $g(z) \equiv \prod_{\nu=1}^{N} (z - \zeta_\nu)$.

PROOF. Assume $N \leqq n - M$, and let $A(z) \equiv \sum_{\nu=0}^{n} a_\nu z^\nu$ be an element of Λ. Let $B(z) \equiv A_0(z)$ if $A_0(z) \not\equiv A(z)$, and let $B(z) \equiv A_0(z) + g(z)Q(z)$ if $A_0(z) \equiv A(z)$. Then $B(z)$ is of degree $\leqq n$, it belongs to Λ, $B(z) \not\equiv A(z)$, and $B(z) = 0$ throughout S. Thus $A(z)$ cannot be an (n, Λ, S) infrapolynomial. This proves (a).

Consider now the two cases: (i) $N = n - M + 1$, and (ii) $N = n - M + 2$, $L_{N-1} = 0$. If $B(z)$ is a polynomial of degree $\leqq n$ belonging to Λ and vanishing throughout S, we can set

$$B(z) \equiv P(z) - l(z)Q(z),$$

where $l(z)$ is a polynomial of degree $\leqq n - M$ which equals $P(z)/Q(z)$ on S. Thus $l(z) \equiv L(z)$ and therefore $B(z) \equiv A_0(z)$. Hence $A_0(z)$ (which is of degree $\leqq n$, belongs to Λ, and vanishes throughout S) is an (n, Λ, S) infrapolynomial. Furthermore, if $A(z)$ is an arbitrary (n, Λ, S) infrapolynomial, then since $A_0(z) = 0$ throughout S, $A(z) \equiv A_0(z)$. Thus we have proved (b), and under the assumption $L_{N-1} = 0$, also (c).

We observe in cases (b) and (c) that if all the values $w_j^{(\nu)}$ are zero, then $P(z) \equiv 0$, $L(z) \equiv 0$, $A_0(z) \equiv 0$, $L_{N-1} = 0$; and 0 is the unique (n, Λ, S) infrapolynomial.

Finally, we consider the case $N = n - M + 2$, $L_{N-1} \neq 0$. We may assume $N > 1$, since if $N = 1$, $P(z)$ is the unique polynomial of degree $\leqq n$ belonging to Λ, and therefore $P(z)$ is the unique (n, Λ, S) infrapolynomial, which proves the assertion of (c).

Let Λ^* denote the set of all polynomials of degree $N-1$ with leading coefficient 1. For every polynomial $A(z)$ of degree $\leqq n$ belonging to Λ, let $A^*(z)$ generically denote the element of Λ^* satisfying

[4] Compare [1961, Theorems 1 and 2].

$$(2)\qquad \begin{aligned} A(z) &\equiv P(z) + [-L(z) + L_{N-1}A^*(z)]Q(z) \\ &\equiv A_0(z) + L_{N-1}A^*(z)Q(z). \end{aligned}$$

For every $B_1(z) \in \Lambda^*$,

$$B(z) \equiv P(z) + [-L(z) + L_{N-1}B_1(z)]Q(z)$$

is of degree $\leqq n$, belongs to Λ, and satisfies $B_1(z) \equiv B^*(z)$.

From (2) we have

$$(3)\qquad A(z) = L_{N-1}A^*(z)Q(z) \qquad \text{throughout } S.$$

From (2) and (3) it follows that a polynomial $A(z)$ of degree $\leqq n$ and belonging to Λ is an (n, Λ, S) infrapolynomial if and only if $A^*(z)$ is an infrapolynomial on S, i.e., [1957, Theorem 13] if and only if $A^*(z)$ is of the form $\sum_{\nu=1}^{N} \lambda_\nu(g(z)/(z-\zeta_\nu))$ $(\lambda_\nu \geqq 0, \sum_{\nu=1}^{N} \lambda_\nu = 1)$. From this, the desired result readily follows.

It may be noted that since $A_0(z)=0$ throughout S, we may set $A_0(z) \equiv P_0(z)g(z)$ where $P_0(z)$ is a polynomial. In case (c), its degree is $\leqq n-N+1$.

5. We turn now to applications of Theorem 1(c). For the case $k=1$, see [1961, Theorems 6b and 10].

THEOREM 2. *Under the conditions of Theorem* 1(c) *let two disjoint circular regions* C_1: $|z-c_1| \leqq r_1$ *and* C_2: $|z-c_2| \leqq r_2$ *(where* $0 \leqq r_1, r_2 < \infty$*) contain, respectively, the set* T *and the zeros of* $A_0(z)$. *Then* C_2 *together with the* $n-N+1$ *circular regions* $|z-(c_1-\epsilon c_2)/(1-\epsilon)| \leqq (r_1+r_2)/|1-\epsilon|$, $\epsilon^{n-N+2}=1$, $\epsilon \neq 1$, *contains all zeros of every* (n, Λ, S) *infrapolynomial.*

PROOF. Let $A(z)$ be an (n, Λ, S) infrapolynomial. By (1) and by the end of §4,

$$A(z) \equiv P_0(z)g(z) + L_{N-1}Q(z) \sum_{\nu=1}^{N} \lambda_\nu \frac{g(z)}{z-\zeta_\nu}\,.$$

The case $L_{N-1}=0$ is trivial; so we assume $L_{N-1} \neq 0$. Suppose that $A(z_0)=0$, $z_0 \notin C_2$. We may set [1950, §1.5.1]

$$\sum_{\nu=1}^{N} \frac{\lambda_\nu}{z_0-\zeta_\nu} = \frac{1}{z_0-\zeta}$$

where $\zeta \in C_2$. Hence

$$(4)\qquad (z_0-\zeta)P_0(z_0) = -L_{N-1}Q(z_0).$$

Also, since $P_0(z)$ is of degree $n-N+1$, and its leading coefficient is $-L_{N-1}$, we have [1922]

$$(z_0 - \zeta)P_0(z_0) = -L_{N-1}(z_0 - \zeta')^{n-N+2} \qquad (\zeta' \in C_2).$$

Similarly, the right-hand side of (4) can be represented in the form $-L_{N-1}(z_0-z')^{n-N+2}$ with $z' \in C_1$. Thus

$$(z_0 - \zeta')^{n-N+2} = (z_0 - z')^{n-N+2}.$$

Since C_1 and C_2 are disjoint, $\zeta' \neq z'$, and therefore

$$z_0 = \frac{z' - \epsilon\zeta'}{1 - \epsilon}, \qquad \epsilon^{n-N+2} = 1, \epsilon \neq 1.$$

Hence [cf. 1949, Lemma (17, 2a)]

$$\left| z_0 - \frac{c_1 - \epsilon c_2}{1 - \epsilon} \right| \leqq \frac{r_1 + r_2}{|1 - \epsilon|}.$$

This completes the proof.

6. If we modify Theorem 2 by assuming that (i) C_1 is $|z-c_1| \geqq r_1$ (and C_2 is as before), or by assuming that (ii) C_2 is $|z-c_2| \geqq r_2$ (and C_1 is as in the theorem) then we may similarly conclude (cf. loc. cit., Lemma (17, 2b)) that every zero z_0 of an (n, Λ, S) infrapolynomial which does not belong to C_2 satisfies

$$\left| z_0 - \frac{c_1 - \epsilon c_2}{1 - \epsilon} \right| \geqq \frac{r_1 - r_2}{|1 - \epsilon|}, \qquad \epsilon^{n-N+2} = 1, \epsilon \neq 1, \text{ in case (i)},$$

and

$$\left| z_0 - \frac{c_1 - \epsilon c_2}{1 - \epsilon} \right| \geqq \frac{r_2 - r_1}{|1 - \epsilon|}, \qquad \epsilon^{n-N+2} = 1, \epsilon \neq 1, \text{ in case (ii)}.$$

7. In the special case of Theorem 2 in which $k=2$, $m_1=m_2=0$, $L_{N-1}\neq 0$, $P_0(z)$ must be of the first degree, say, $P_0(z) \equiv -L_{N-1}(z-a)$, and (4) may be written as

$$(z_0 - \zeta)(z_0 - a) = (z_0 - z_1)(z_0 - z_2), \qquad a \in C_2.$$

Thus $z_0(z_1+z_2-a-\zeta) = z_1z_2 - a\zeta$. From the fact that C_1 and C_2 are disjoint, one easily infers that the coefficient of z_0 in the last equality is not zero. Hence

$$z_0 = \frac{z_1z_2 - a\zeta}{z_1 + z_2 - a - \zeta}.$$

Thus z_0 lies in the image of C_2 under the linear transformation $\phi(z) \equiv (z_1z_2 - az)/(z_1+z_2-a-z)$, and this image can be readily determined in terms of c_2, r_2, z_1, z_2 and a.

8. THEOREM 3. *Let the hypotheses of Theorem* 1(c) *hold with* $L_{N-1}\neq 0$, *and suppose also that all the* m_j *are zero. Set*

$$P_0(z) \Big/ \left\{ L_{N-1} \prod_{\nu=1}^{k} (z - z_\nu) \right\} \equiv \sum_{\nu=1}^{k} \frac{b_\nu}{z - z_\nu} \cdot \tag{5}$$

Let z_0 *be a zero of an* (n, Λ, S) *infrapolynomial* $A(z)$.

(a) *If all the* b_ν *are* $\leqq 0$, *then* z_0 *cannot lie on a line or circle L which separates T from S.*

(b) *Suppose* $b_j \leqq 0$ *for* $j=1, 2, \cdots, \mu(<k)$ *and* $b_j \geqq 0$ *for* $j=\mu+1, \mu+2, \cdots, k$. *Then* z_0 *cannot lie on a line or circle L which separates* T_1 *from* $S \cup T_2$. *Here* $T_1 = \{z_1, z_2, \cdots, z_\mu\}$, $T_2 = \{z_{\mu+1}, z_{\mu+2}, \cdots, z_k\}$.

Note that $\sum_{\nu=1}^{k} b_\nu = -1$.

COROLLARY. *Under the hypotheses of Theorem* 3(a), *if* $T \subseteq D_1$ *and* $S \subseteq D_2$ *where* D_1 *and* D_2 *are disjoint segments of a line or disjoint arcs of a circle, then* $z_0 \in D_1 \cup D_2$. *Likewise, under the hypotheses of Theorem* 3(b), *if* $T_1 \subseteq D_1$ *and* $S \cup T_2 \subseteq D_2$ *where* D_1 *and* D_2 *are as before, then* $z_0 \in D_1 \cup D_2$.

[Compare 1950, §4.2.3., Theorem 2].

Theorem 3 follows by the method of the first paragraph of [1950, §4.2.1.].

9. We shall now suppose that S and T are symmetric in the x-axis.

THEOREM 4. *Let the hypotheses* (*preceding* (a)) *of Theorem* 3 *hold, and suppose both S and T are symmetric in the x-axis, T has no point on the y-axis, and S lies in* $\mathrm{Re}(z) > 0$. *Suppose that* $A(z)$ *is a real polynomial, that* $b_\nu \leqq 0$ *whenever* $\mathrm{Re}(z_\nu) < 0$ *and* $b_\nu \geqq 0$ *whenever* $\mathrm{Re}(z_\nu) > 0$, *and that* $b_i = b_j$ *whenever* $z_i = \bar{z}_j$. *Then if* z_0 *is nonreal, there exists* (*at least one*) *nonreal point* ξ *of* $S \cup T$ *such that* z_0 *lies on or within the circle tangent to the line* 0ξ *at* ξ *and to* $0\bar{\xi}$ *at* $\bar{\xi}$.

Theorem 4 and further results can be proved by the methods of [1955] and [1961a]; compare also Marden [1949], who makes a special study of the zeros of rational functions such as the second member of (5).

BIBLIOGRAPHY

1922 J. L. Walsh, *On the location of the roots of certain types of polynomials*, Trans. Amer. Math. Soc. 24 (1922), 163–180.

1935 ———, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloq. Publ. Vol. 20, Amer. Math. Soc., Providence, R. I., 1935.

1949 M. Marden, *The geometry of the zeros of a polynomial in a complex variable*, Mathematical Surveys, No. 3. Amer. Math. Soc., Providence, R. I., 1949.

1950 J. L. Walsh, *The location of critical points of analytic and harmonic functions*, Amer. Math. Soc. Colloq. Publ. Vol. 34, Amer. Math. Soc., Providence, R. I., 1950.

1955 ———, *A generalization of Jensen's theorem on the zeros of the derivative of a polynomial*, Amer. Math. Monthly **62** (1955), 91–93.

1956 J. L. Walsh and M. Zedek, *On generalized Tchebycheff polynomials*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 99–104.

1957 T. S. Motzkin and J. L. Walsh, *Underpolynomials and infrapolynomials*, Illinois J. Math. **1** (1957), 406–426.

1961 O. Shisha and J. L. Walsh, *The zeros of infrapolynomials with some prescribed coefficients*, J. Analyse Math. **9** (1961), 111–160.

1961a J. L. Walsh, *A new generalization of Jensen's theorem on the zeros of the derivative of a polynomial*, Amer. Math. Monthly **68** (1961), 978–983.

1962 O. Shisha, *An extension of Jensen's theorem for the derivative of a polynomial and for infrapolynomials*, J. Res. Nat. Bur. Standards Sect. B **66** (1962), no. 2, 53–55.

HARVARD UNIVERSITY AND
NATIONAL BUREAU OF STANDARDS