# ON A THEOREM OF P. J. COHEN AND H. DAVENPORT

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Let $G$ be a compact Abelian group with character group X. P. J. Cohen [1] has proved that if X is torsion-free, if $\Upsilon$ is a finite subset of X consisting of $N$ characters, and $|\alpha_\chi| \geqq 1$ for all $\chi \in \Upsilon$, then

$$(0) \qquad \int_G \left| \sum_{\chi \in \Upsilon} \alpha_\chi \chi(x) \right| dx > K \left( \frac{\log N}{\log \log N} \right)^{1/8},$$

where the integral is the Haar integral on $G$, $K$ is some positive constant not depending on $G$, and $N$ is sufficiently large. For the case in which $G$ is the circle group, H. Davenport [2] has improved (0) by replacing the exponent $\frac{1}{8}$ by $\frac{1}{4}$ and the constant $K$ by $\frac{1}{8}$. Cohen's and Davenport's arguments can in all likelihood be combined to yield (0) with exponent $\frac{1}{4}$ for an arbitrary $G$ such that X is torsion-free.

In this note we apply Davenport's ideas to prove (0) with exponent $\frac{1}{4}$ not only for the case of torsion-free X but also for the case in which the torsion subgroup of X is an arbitrary finite Abelian group. By using care in our estimates we find some fairly large possible $K$'s, and we also work out some numerical cases. If X has infinite torsion subgroup, we show that no inequality like (0) can possibly hold.

THEOREM A. *Let $G$ be a compact Abelian group with character group* X. *Suppose that the torsion subgroup of* X *is finite and consists of $f$ elements. Let $\Upsilon$ be a set of $N$ distinct elements of* X, *where $N$ is a positive integer. For each $\chi \in \Upsilon$, let $\alpha_\chi$ be a complex number such that $|\alpha_\chi| \geqq 1$. Then for every number $K < (1-e^{-2})6^{-1/2}$, we have*

$$(1) \qquad \int_G \left| \sum_{\chi \in \Upsilon} \alpha_\chi \chi(x) \right| dx > K \left( \frac{\log N}{\log \log N} \right)^{1/4}$$

*provided that $N$ is sufficiently large, depending upon $K$ and $f$. For example, if $K = 3/10$, (1) holds for all $N$ such that either*

$$(2) \qquad \begin{aligned} &N > e^{310} \quad \textit{and} \quad f \leqq 6 \textit{ or} \\ &N \geqq \left( \frac{3}{2} f^2 \right)^{3f^2} \quad \textit{and} \quad f > 6. \end{aligned}$$

Received by the editors July 30, 1962.

[1] The research of both authors was supported by the National Science Foundation, under grant NSF G-18838.

PROOF. Throughout the proof, we suppose that

$$N > f. \tag{3}$$

Let $\Phi$ be the subgroup of X generated by $\Upsilon$. Then $\Phi$ is isomorphic with a direct product of finitely many cyclic groups, say $a$ infinite cyclic groups and $b$ finite cyclic groups. Thus every $\chi \in \Upsilon$ corresponds to a unique sequence

$$(c_1, c_2, \cdots, c_a, c_{a+1}, \cdots, c_{a+b}), \tag{4}$$

where $c_1, c_2, \cdots, c_a$ are arbitrary integers and $c_{a+1}, c_{a+2}, \cdots, c_{a+b}$ are nonnegative integers less than some fixed positive integers. We have $b \geqq 0$ and $a > 0$ since $N > f$.

We impose a complete ordering on $\Phi$. If $\chi$ and $\chi'$ are distinct elements of $\Phi$, if $\chi$ corresponds to $(c_1, c_2, \cdots, c_{a+b})$, and $\chi'$ corresponds to $(c_1', c_2', \cdots, c_{a+b}')$, then we write $\chi < \chi'$ if $c_j < c_j'$ where $j$ is the first index $k$ for which $c_k$ differs from $c_k'$. We now write $\Upsilon$ as

$$\{\chi_1, \chi_2, \cdots, \chi_N\} \qquad \text{where } \chi_1 < \chi_2 < \cdots < \chi_N. \tag{5}$$

For $\chi \in \Phi$, let $N(\chi)$ be the cardinal number of the set $\{\psi \in \Upsilon : \psi \leqq \chi\}$.

Following Cohen's construction, we now define subsets $P_0, P_1, \cdots, P_k$ of $\Phi$ and subsets $T_0, T_1, \cdots, T_k$ of $\Upsilon$, where $k$ is a positive integer to be chosen later. Each set $T_j$ is to consist of exactly $r$ characters, where $r$ is an integer $>1$. First, we set $P_0 = \{\chi_1\}$ and $T_0 = \varnothing$. Suppose that the sets $P_0, P_1, \cdots, P_{l-1}$ and the sets $T_0, T_1, \cdots, T_{l-1}$ have been defined. We determine the elements $\chi_{m_1^{(l)}}, \chi_{m_2^{(l)}}, \cdots, \chi_{m_r^{(l)}}$ of $T_l$ as follows. We take $m_1^{(l)} = 1$. Suppose that the indices $m_1^{(l)}, m_2^{(l)}, \cdots, m_{j-1}^{(l)}$ have been determined ($j \leqq r$). Then we want $m_j^{(l)}$ to be the smallest index $m > m_{j-1}^{(l)}$ such that

$$\chi \chi_{m_i^{(l)}} \bar{\chi}_m \notin \Upsilon \tag{6}$$

for all $\chi \in P_{l-1}$ and all $i = 1, 2, \cdots, j-1$. Let us find conditions under which $m$ exists.

Let $\chi$, $\chi'$, and $\chi''$ be any characters in $\Phi$, and let $(c_1, c_2, \cdots, c_{a+b})$, $(c_1', c_2', \cdots, c_{a+b}')$, $(c_1'', c_2'', \cdots, c_{a+b}'')$ be the corresponding sequences (4). Now suppose that $\chi'$ and $\chi''$ are such that $\chi' < \chi''$ and that $q$ is the smallest index such that $c_q'$ differs from $c_q''$. If $q$ is less than or equal to $a$, then the inequality

$$\chi \chi' \bar{\chi}'' < \chi \tag{7}$$

obtains. We restrict $m$ to be greater than or equal to $m_{j-1}^{(l)} + f$. Thus the inequality (7) applies to the product appearing in (6). Among these $m$'s, we count those which must be rejected for violating (6).

For each $i$, we have to reject $\chi_m$ if

$$\chi_m = \chi\chi_{m_i^{(l)}}\bar{\psi}$$

where $\bar{\psi}\in\Upsilon$ and $\chi\in P_{l-1}$. Such a $\psi$ must be less than $\chi$ in the ordering (5), and so we reject at most

$$\sum_{\chi\in P_{l-1}} N(\chi)$$

characters for each $i$. Consequently, taking $i=1, 2, \cdots, j-1$, we reject at most

$$(j-1)\sum_{\chi\in P_{l-1}} N(\chi)$$

characters. It follows that $m_j^{(l)}$ exists if

$$m_{j-1}^{(l)} + (f-1) + (j-1)\sum_{\chi\in P_{l-1}} N(\chi) \leqq N, \tag{8}$$

and that

$$m_j^{(l)} \leqq m_{j-1}^{(l)} + (f-1) + (j-1)\sum_{\chi\in P_{l-1}} N(\chi) \tag{9}$$

if (8) holds. Supposing that (8) holds for $j=r$ (and hence for $j=2, 3, \cdots, r-1$), we sum (9) over $j=2, 3, \cdots, i$ $(2\leqq i\leqq r)$ to obtain

$$m_i^{(l)} \leqq 1 + (i-1)(f-1) + \frac{i(i-1)}{1}\sum_{\chi\in P_{l-1}} N(\chi). \tag{10}$$

The inequality (10) also holds for $i=1$.

We next define the set $P_l\subset\Phi$:

$$P_l = P_{l-1}\cup\left\{\chi\chi_{m_i^{(l)}}\bar{\chi}_{m_j^{(l)}}\colon \chi\in P_{l-1};\ 1\leqq i<j\leqq r\right\}\cup T_l. \tag{11}$$

The right side of (10) is obviously monotonic increasing in $i$ and is also monotonic nondecreasing in $l$ because $P_{l-1}\subset P_l$. Hence we can find the sets of characters $T_1, T_2, \cdots, T_k$ provided that

$$1 + (r-1)(f-1) + \frac{r(r-1)}{2}\sum_{\chi\in P_{k-1}} N(\chi) \leqq N. \tag{12}$$

Let us now use (11) to estimate the size of $\sum_{\chi\in P_{k-1}} N(\chi)$. For $l=1, 2, \cdots, k-1$, (11) implies that

$$\sum_{\chi\in P_l} N(\chi) \leqq \sum_{\chi\in P_{l-1}} N(\chi) + \sum{}' N(\chi\chi_{m_i^{(l)}}\bar{\chi}_{m_j^{(l)}}) + \sum_{\chi\in T_l} N(\chi), \tag{13}$$

where $\sum'$ denotes the sum over $\chi\in P_{l-1}$ and $1\leqq i<j\leqq r$. We chose

$m_1^{(l)}, m_2^{(l)}, \cdots, m_r^{(l)}$ in such a way that the relation (7) holds for the product $\chi\chi_{m_i^{(l)}}\bar{\chi}_{m_j^{(l)}}$. Hence we have

$$(14)\qquad \sum{}' N(\chi\chi_{m_i^{(l)}}\bar{\chi}_{m_j^{(l)}}) \leqq \sum{}' N(\chi) \leqq \frac{r(r-1)}{2}\sum_{\chi\in \mathrm{P}_{l-1}} N(\chi).$$

For $\chi_{m_i^{(l)}}\in \mathrm{T}_l$, it is obvious that $N(\chi_{m_i^{(l)}})=m_{i^{(l)}}$. Using this and (10), we write

$$(15)\qquad \begin{aligned}\sum_{\chi\in \mathrm{T}_l} N(\chi) &= \sum_{i=1}^{r} m_i^{(l)}\\ &\leqq \sum_{i=1}^{r}\left\{1+(i-1)(f-1)+\frac{i(i-1)}{2}\sum_{\chi\in \mathrm{P}_{l-1}} N(\chi)\right\}\\ &= r+\frac{r(r-1)}{2}(f-1)+\frac{r(r^2-1)}{6}\sum_{\chi\in \mathrm{P}_{l-1}} N(\chi).\end{aligned}$$

Combining (13), (14), and (15), we see that

$$(16)\qquad \begin{aligned}\sum_{\chi\in \mathrm{P}_l} N(\chi) \leqq r&+\frac{r(r-1)}{2}(f-1)\\ &+\frac{r^3+3r^2-4r+6}{6}\sum_{\chi\in \mathrm{P}_{l-1}} N(\chi).\end{aligned}$$

The recurrence inequality (16) has the form

$$A_l \leqq a + bA_{l-1}, \qquad b>1, \quad a>0.$$

This implies that

$$(17)\qquad \begin{aligned}A_l &\leqq b^l\left(A_0+\frac{a}{b-1}\right)-\frac{a}{b-1}\\ &\leqq b^l\left(1+\frac{a}{b-1}\right).\end{aligned}$$

From now on, suppose that

$$(18)\qquad r \geqq \max(6, f) \quad \text{and} \quad k \text{ is an integer } \geqq 3.$$

The inequality (17) for $l=k-1$ is

$$\sum_{\chi\in \mathrm{P}_{k-1}} N(\chi) = A_{k-1} \leqq b^{k-1}\left(1+\frac{a}{b-1}\right),$$

where

$$b = \frac{r^3 + 3r^2 - 4r + 6}{6} \leqq \frac{1}{4} r^3,$$

$$b - 1 \geqq \frac{r^3}{6},$$

$$a = r + \frac{r(r-1)}{2}(f-1) \leqq fr^2.$$

Thus we have

$$\sum_{\chi \in \mathrm{P}_{k-1}} N(\chi) \leqq \left(\frac{r^3}{4}\right)^{k-1}\left(1 + \frac{6}{r^3} fr^2\right)$$

$$= \left(\frac{r^3}{4}\right)^{k-1}\left(1 + \frac{6f}{r}\right) \leqq r^{3(k-1)} \frac{7}{4^{k-1}} < r^{3(k-1)}.$$

We now return to relation (12). A routine computation shows that (12) holds if

$$(19) \qquad r^{3k-1} \leqq N.$$

Thus we can find the sets of characters $\mathrm{T}_1, \mathrm{T}_2, \cdots, \mathrm{T}_k$ provided that (18) and (19) hold.

In determining $k$ and $r$ for which (19) holds, it is convenient to take $k = r^2$; and this choice of $k$ turns out to be satisfactory for the final arguments of the present proof. Then we take

$$(20) \qquad r = \left[\left(\frac{2 \log N}{3 \log \log N}\right)^{1/2}\right],$$

where $[\cdots]$ is the greatest integer symbol and

$$(21) \qquad N \geqq e^e.$$

It is then easy to see that (19) holds.

We next construct by induction a sequence of functions $\phi_0, \phi_1, \cdots, \phi_k$ on the group $G$. Each $\chi$ in $\Upsilon$ is a certain $\chi_j$ under the ordering of (5); we write $\beta_j = \alpha_{\chi_j} |\alpha_{\chi_j}|^{-1}$. We define $\phi_0$ to be the function $\beta_1 \chi_1$. For $l \in \{1, 2, \cdots, k\}$, suppose that $\phi_{l-1}$ has been defined. Following Davenport [2], we define $\phi_l$ by the equality

$$(22) \qquad \begin{aligned} \phi_l = \phi_{l-1}&\left\{1 - \frac{2}{r^2} - \frac{1}{r^3} \sum{}' \beta_{m_i^{(l)}} \chi_{m_i^{(l)}} \bar{\beta}_{m_j^{(l)}} \bar{\chi}_{m_j^{(l)}}\right\} \\ &+ \frac{1}{r^{5/2}} \sum_{j=1}^{r} \beta_{m_j^{(l)}} \chi_{m_j^{(l)}}; \end{aligned}$$

in which the sum $\sum'$ is over all $i, j$ such that $1 \leqq i < j \leqq r$.

Applying the method of Davenport's Lemma 2 [2], we make the following computations. First, for $x \in G$, write

$$\sum{}' \beta_{m_i^{(l)}} \bar{\beta}_{m_j^{(l)}} \chi_{m_i^{(l)}}(x) \bar{\chi}_{m_j^{(l)}}(x) = P + iQ.$$

Then it is clear that

$$P^2 + Q^2 \leqq \left(\frac{1}{2} r(r-1)\right)^2 < \frac{1}{4} r^4$$

and

$$\left| \sum_{j=1}^{r} \beta_{m_j^{(l)}} \chi_{m_j^{(l)}}(x) \right|^2 = r + 2 \operatorname{Re} \sum{}' \beta_{m_i^{(l)}} \bar{\beta}_{m_j^{(l)}} \chi_{m_i^{(l)}}(x) \bar{\chi}_{m_j^{(l)}}(x)$$
$$= r + 2P.$$

Hence we have

$$P \geqq \frac{r}{2} \cdot$$

Since $r \geqq 6$, Davenport's Lemma 1 shows that if $|\phi_{l-1}(x)| \leqq 1$, then

$$|\phi_l(x)| \leqq \left| 1 - \frac{2}{r^2} - \frac{P + iQ}{r^3} \right| + \frac{1}{r^{5/2}} (r + 2P)^{1/2} \leqq 1.$$

Since $|\phi_0| = |\beta_1 \chi_1| = 1$, it follows that $|\phi_l| \leqq 1$ $(l = 0, 1, \cdots, k)$.

We also follow Davenport to define

$$I_l = \int_G \phi_l(x) \sum_{\chi \in \mathrm{T}} \bar{\alpha}_\chi \bar{\chi}(x) dx. \tag{23}$$

The construction of $\mathrm{T}_1, \mathrm{T}_2, \cdots, \mathrm{T}_k$ shows that

$$I_l = \left(1 - \frac{2}{r^2}\right) I_{l-1} + \frac{1}{r^{5/2}} \sum_{j=1}^{r} \beta_{m_j^{(l)}} \bar{\alpha}_{\chi_{m_j^{(l)}}}$$
$$= \left(1 - \frac{2}{r^2}\right) I_{l-1} + \frac{1}{r^{5/2}} \sum{}' |\alpha_\chi|,$$

where $\sum'$ denotes the sum over $\chi = \chi_{m_j^{(l)}}$, $j = 1, 2, \cdots, r$. Since $|\alpha_\chi| \geqq 1$, we now have

$$I_l \geqq \left(1 - \frac{2}{r^2}\right) I_{l-1} + \frac{1}{r^{3/2}} \qquad (l = 1, 2, \cdots, k-1). \tag{24}$$

Thus we have

$$I_l - \frac{1}{2} r^{1/2} \geqq \left(1 - \frac{2}{r^2}\right)\left(I_{l-1} - \frac{1}{2} r^{1/2}\right),$$

so that

$$I_k - \frac{1}{2} r^{1/2} \geqq \left(1 - \frac{2}{r^2}\right)^k \left(I_0 - \frac{1}{2} r^{1/2}\right),$$

$$I_k \geqq \frac{1}{2} r^{1/2} - \left(1 - \frac{2}{r^2}\right)^k \left(\frac{1}{2} r^{1/2} - |\alpha_{\chi_1}|\right),$$

and

$$I_k \geqq \frac{1}{2} r^{1/2} - \left(1 - \frac{2}{r^2}\right)^k \left(\frac{1}{2} r^{1/2} - 1\right). \tag{25}$$

If $t > 2^{1/2}$, then

$$\left(1 - \frac{2}{t^2}\right)^{t^2} < \frac{1}{e^2} \cdot$$

hence for $k = r^2$, we have

$$I_k > \frac{1}{2} (1 - e^{-2}) r^{1/2}. \tag{26}$$

Since $|\phi_k| \leqq 1$, we have

$$\begin{aligned} \int_G \left| \sum_{\chi \in \Upsilon} \alpha_\chi \chi(x) \right| dx &\geqq \left| \int_G \phi_k(x) \sum_{\chi \in \Upsilon} \bar{\alpha}_\chi \bar{\chi}(x) dx \right| \\ &= I_k > \frac{1}{2} (1 - e^{-2}) r^{1/2}. \end{aligned} \tag{27}$$

Using the value for $r$ in (20), we obtain

$$\begin{aligned} \frac{1}{2} (1 - e^{-2}) r^{1/2} &> \frac{1}{2} (1 - e^{-2}) \left(\left(\frac{2 \log N}{3 \log \log N}\right)^{1/2} - 1\right)^{1/2} \\ &= (1 - e^{-2}) 6^{-1/2} \left(1 - \left(\frac{3 \log \log N}{2 \log N}\right)^{1/2}\right)^{1/2} \left(\frac{\log N}{\log \log N}\right)^{1/4}. \end{aligned} \tag{28}$$

Now let $K$ be any number such that $K < (1 - e^{-2}) 6^{-1/2}$. Combining (27) and (28), we see that

$$\int_G \left| \sum_{\chi \in \Upsilon} \alpha_\chi \chi(x) \right| dx > K \left(\frac{\log N}{\log \log N}\right)^{1/4} \tag{29}$$

for $N$ so large that conditions (3), (18), (21) hold and

$$(1 - e^{-2})6^{-1/2}\left(1 - \left(\frac{3 \log \log N}{2 \log N}\right)^{1/2}\right)^{1/2} \geqq K. \tag{30}$$

Note that $r$ is defined by (20) and that $k=r^2$.

The condition $k\geqq 3$ is implied by $r\geqq 6$. The conditions (3) and (21) will be trivially satisfied by the following. Since 6 and $f$ are integers, (18) will be satisfied if

$$\frac{\log N}{\log \log N} \geqq \frac{3}{2} \max (36, f^2) = \max\left(54, \frac{3}{2} f^2\right). \tag{31}$$

If $K$ is taken less than $((1-e^{-2})/6)5^{1/2}$, say $K=3/10$, then (30) is implied by (31). To see how large $N$ must be for (31) to hold, define $u$ by

$$N = e^{2u \log u}.$$

Then

$$\frac{\log N}{\log \log N} = 2u\left(\frac{\log 2}{\log u} + 1 + \frac{\log \log u}{\log u}\right)^{-1} \geqq u,$$

since $\log \log u \leqq \log u - 1$. Hence if $N\geqq (3f^2/2)^{3f^2}$, we have

$$\frac{\log N}{\log \log N} \geqq \frac{3}{2} f^2.$$

Furthermore, if $N>e^{310}$, then $\log N/\log \log N>54$. Thus if $f\leqq 6$, (29) holds with $K=3/10$ for all $N>e^{310}$. The inequality (29) with $K=3/10$ also holds for $N\geqq(3f^2/2)^{3f^2}$ if $f>6$. This completes the proof of Theorem A.

THEOREM B. *Let $G$ be a compact Abelian group with character group $X$, and let $\Gamma$ be a finite subgroup of $X$. Then*

$$\int_G \left| \sum_{\chi\in\Gamma} \chi(x) \right| dx = 1. \tag{32}$$

PROOF. Let $A=\{x\in G: \chi(x)=1 \text{ for all } \chi\in\Gamma\}$. It is well known that the quotient group $G/A$ is isomorphic with the character group of $\Gamma$. Let $\lambda$ denote normalized Haar measure on $G$. Then we have $\lambda(A)=1/o(\Gamma)$. For $\chi_0\in\Gamma$ and $x\in G$, we have

$$\sum_{\chi\in\Gamma} \chi(x) = \sum_{\chi\in\Gamma} \chi_0\chi(x) = \chi_0(x) \sum_{\chi\in\Gamma} \chi(x).$$

If $\sum_{\chi\in\Gamma} \chi(x)\neq 0$, it follows that $\chi_0(x)=1$. Since $\chi_0$ can be any element of $\Gamma$, we have $\sum_{x\in\Gamma}\chi(x)=0$ if $x\notin A$. It is trivial that $\sum_{x\in\Gamma}\chi(x)=o(\Gamma)$ if $x\in A$. The equality (32) follows at once.

Theorem B shows that the number $N$ appearing in Theorem A must go to infinity as $f$ goes to infinity. It also shows that Theorem A fails completely if the torsion subgroup of X is infinite. For in this case X contains finite subgroups of arbitrarily large order, and (32) shows that nothing like (1) can hold.

It should also be noted that if the torsion subgroup of X is finite, then it is a direct factor of X. First, the torsion subgroup is always a pure subgroup of X; and then one can quote, for example, a well-known theorem of Łoś [3, (25.21)]. Thus $G$ is topologically the union of a finite number of replicas of a connected compact Abelian group. There appears to be no advantage in using this fact for our proof.

## Bibliography

1. P. J. Cohen, *On a conjecture of Littlewood and idempotent measures*, Amer. J. Math. 82 (1960), 191–212.

2. H. Davenport, *On a theorem of P. J. Cohen*, Mathematika 7 (1960), 93–97.

3. Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis*, Vol. I, Grundlehren der Math. Wiss., Band 115, Springer-Verlag, Heidelberg, 1963.

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