WEAKLY COMPACT $B^{\#}$-ALGEBRAS

A. OLUBUMMO

1. A complex Banach algebra A is a compact (weakly compact) algebra if its left and right regular representations consist of compact (weakly compact) operators. Let E be any subset of A and denote by E_l and E_r the left and right annihilators of E. A is an annihilator algebra if $A_l = (0) = A_r$, $I_r \neq (0)$ for each proper closed left ideal I and $J_l \neq (0)$ for each proper closed right ideal J.

In [6, Theorem 1], it was shown that a semi-simple compact algebra is an annihilator algebra. The first main result of the present paper (Theorem 2.1) is that a semi-simple annihilator algebra is a weakly compact algebra. Thus if $\mathcal{C}$, $\mathcal{A}$, $\mathcal{W}$ denote respectively the class of all semi-simple compact algebras, all semi-simple annihilator algebras and all weakly compact algebras, we have $\mathcal{C} \subset \mathcal{A} \subset \mathcal{W}$.

§3 is devoted to the structure theory of weakly compact $B^{\#}$-algebras begun in [7]. A Banach algebra A is a $B^{\#}$-algebra if, given $a \in A$, there exists $a^{\#} \neq 0$ in A such that

$$\|a^{\#}\| \, \|a\| = \|(a^{\#}a)^n\|^{1/n}, \qquad n = 1, 2, 3, \cdots .$$

In their study of weakly compact B^*-algebras Ogasawara and Yoshinaga [4] obtained the following structure theorem:

THEOREM. *The following statements are equivalent for a B^*-algebra A*:

(1) *A is a weakly compact algebra*;

(2) *A is the $B^*(\infty)$-sum of C^*-algebras, each of which consists of the set of all compact operators on a Hilbert space.*

The following result was obtained in [7, Theorem 3.1]:

THEOREM. *A Banach algebra A is the algebra $F(X)$ of all uniform limits of operators of finite rank on a reflexive Banach space X if and only if A is a simple, weakly compact $B^{\#}$-algebra with minimal ideals.*

Making use of this result and our present Theorem 2.1, we now obtain the following more general result:

THEOREM 3.4. *The following statements are equivalent*:

(1) *A is a weakly compact $B^{\#}$-algebra with a dense socle*;

(2) *A is the $B(\infty)$-sum of $B^{\#}$-algebras, each of which is the algebra of all uniform limits of operators of finite rank on a reflexive Banach space.*

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We note that every B^*-algebra is a $B^\#$-algebra and that a weakly compact B^*-algebra automatically has a dense socle [4, p. 15], so that Theorem 3.4 includes the result of Ogasawara and Yoshinaga.

2. THEOREM 2.1. *A semi-simple annihilator algebra A is a weakly compact algebra.*

PROOF. Let Ae, $e^2=e$, be a minimal left ideal of A. Then Ae is a reflexive Banach space since it is also a minimal left ideal of the simple annihilator algebra $(AeA)^-$ [1, Theorem 13]. Let $a\in A$; then [3, p. 483, Corollary 3] right multiplication by ae is a weakly compact mapping of A into Ae, and a fortiori, of A into A. Then by [3, p. 484, Theorem 5], right multiplication by a socle element is weakly compact. Since the socle is dense [1, Theorem 4], it follows [3, p. 483, Corollary 4] that any $x\in A$ is a right (and similarly left) weakly compact element.

3. In this section we prove a structure theorem for weakly compact $B^\#$-algebras.

LEMMA 3.1. *Let A be a semi-simple Banach algebra with a dense socle. Then every maximal regular left ideal M of A has a nonzero right annihilator.*

PROOF. If $\{Ae_\alpha\}_{\alpha\in\Omega}$ denotes the set of all the minimal left ideals of A, then there exists $\alpha_0\in\Omega$ such that $Ae_{\alpha_0}\not\subset M$. Further, $M\cap Ae_{\alpha_0}=(0)$ and $M\oplus Ae_{\alpha_0}=A$. Since M is a regular left ideal of A, there exists $j\in A$ such that $xj-x\in M$ for every $x\in A$. For some $a_0\in A$ and $m_0\in M$, we have $j=m_0+a_0e_{\alpha_0}$, $a_0e_{\alpha_0}\neq 0$. Let m be an arbitrary element of M; then $mj-m\in M$ and $mj-ma_0e_{\alpha_0}=mm_0\in M$, from which it follows that $m-ma_0\cdot e_{\alpha_0}\in M$, and therefore $ma_0\cdot e_{\alpha_0}\in M$. However, $ma_0\cdot e_{\alpha_0}\in Ae_{\alpha_0}$ since Ae_{α_0} is a left ideal; thus $ma_0\cdot e_{\alpha_0}\in M\cap Ae_{\alpha_0}=(0)$, and since m is arbitrary in M, the lemma is proved.

LEMMA 3.2. *Let A be a $B^\#$-algebra with a dense socle. If $|\cdot|$ is any norm in A with $|a|\leqq\|a\|$ for each $a\in A$, then $|\cdot|=\|\cdot\|$.*

PROOF. Suppose that $j\in A$ and j has no left reverse. We show that there exists $a\neq 0$ such that $ja=a$. In fact, let $J=[yj-y: y\in A]$; then J is a regular left ideal of A which is proper since $j\notin J$. Now J is contained in a maximal regular left ideal M and by Lemma 3.1, there exists $a\in A$, $a\neq 0$ such that $Ja=(0)$, i.e. such that $yja-ya=0$ for all $y\in A$; i.e., $A(ja-a)=(0)$. Since A, being a $B^\#$-algebra is semi-simple, $A_r=(0)$ from which it follows that $ja=a$. The conclusion now follows exactly as in [2, Theorems 3 and 4].

LEMMA 3.3. *A semi-simple Banach algebra A with a dense socle (or with the annihilator property) is the completion of the direct join of all its minimal closed two-sided ideals.*

This is essentially Theorem 6 of Bonsall and Goldie [1], under the hypothesis that A be an annihilator algebra. The annihilator property implies that A has a dense socle and this, together with the semi-simplicity of A, is all that is required to prove the theorem.

DEFINITION. Let $\{A_\alpha\}_{\alpha\in\Omega}$ denote a set of Banach algebras. The $B(\infty)$-sum of the A_α is the Banach algebra A consisting of all the functions $f(\cdot)$ defined on Ω with $f(\alpha)\in A_\alpha$ for each $\alpha\in\Omega$ and such that, given $\epsilon>0$, there is a finite subset $\alpha_1, \alpha_2, \cdots, \alpha_n$ of Ω such that $\|f(\alpha)\|<\epsilon$ for $\alpha\neq\alpha_1, \alpha_2, \cdots, \alpha_n$. We define the algebraic operations in A in the obvious way, e.g. $(f+g)(\alpha)=f(\alpha)+g(\alpha)$, etc. and define the norm by $\|f(\cdot)\|=\sup_{\alpha\in\Omega}\|f(\alpha)\|$.

We now state our second main result:

THEOREM 3.4. *The following statements are equivalent:*

(1) *A is a weakly compact $B^\#$-algebra with a dense socle.*

(2) *A is the $B(\infty)$-sum of $B^\#$-algebras A_α, $\alpha\in\Omega$, each of which is the algebra of all uniform limits of operators of finite rank on a reflexive Banach space.*

PROOF. (1) $\Rightarrow$ (2). Bonsall [2, Theorem 6] has shown that if A is a $B^\#$-annihilator algebra, then A is isomorphic and isometric to the $B(\infty)$-sum of its minimal closed two-sided ideals A_α. For the case of a $B^\#$-algebra with a dense socle, Bonsall's proof applies almost word for word if Lemmas 3.2 and 3.3 are borne in mind.

That each A_α is a weakly compact algebra in its own right is clear and that A_α is simple follows readily from a routine argument which depends essentially on the fact that A is semi-simple. Thus each A_α is a simple, weakly compact, $B^\#$-algebra with minimal ideals. (That A_α contains a minimal left ideal of its own follows from the fact that A_α contains a minimal left ideal of A which is also a minimal left ideal of A_α.) Hence by [7, Theorem 3.1], each A_α is the algebra of all uniform limits of operators of finite rank on a reflexive Banach space.

(2) $\Rightarrow$ (1). Each A_α, being the algebra of all uniform limits of operators of finite rank on a reflexive Banach space, is a $B^\#$ annihilator algebra [2, Theorem 2]. Since a $B^\#$-algebra is semi-simple, the $B(\infty)$-sum of the A_α is, by a result of Rickart's [8, p. 107], an annihilator algebra. That the $B(\infty)$-sum of an arbitrary class of $B^\#$-algebras is a $B^\#$-algebra is proved in [5, Lemma 4.7]. Thus the $B(\infty)$-sum A of the A_α is a semi-simple annihilator algebra. From this it follows that

A has a dense socle and by Theorem 2.1, A is weakly compact. This concludes the proof.

COROLLARY. *A $B^{\sharp}$-algebra is an annihilator algebra if and only if it has a dense socle and is a weakly compact algebra.*

It is to be noted that one-handed weak complete continuity is enough to prove (1) $\Rightarrow$ (2). In fact, a very slight modification of the proofs of [7, Lemma 3.1 and Theorem 3.1] shows that a simple, left weakly compact $B^{\sharp}$-algebra with minimal ideals is isomorphic and isometric to the algebra $F(X)$ of all uniform limits of operators of finite rank on a reflexive Banach space X. Since the $B(\infty)$-sum of the A_α in Theorem 3.4 is an annihilator algebra and weakly compact, we obtain the following:

THEOREM 3.5. *A right weakly compact $B^{\sharp}$-algebra with a dense socle is a weakly compact algebra.*

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UNIVERSITY COLLEGE, IBADAN, NIGERIA