ON ODD PERFECT NUMBERS. II

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One of the oldest unsolved mathematical problems is the following one: Are there odd perfect numbers? So many interesting necessary conditions for an odd integer to be perfect have been found out. A bibliography of previous work is given by McCarthy [5].

Throughout this paper n denotes an odd perfect number.

The following results have been proved in [1] and [2] respectively:

(i) $1/2 < \sum_{p|n} (1/p) < 2 \log (\pi/2)$ $(\sim \cdot 903)$,

(ii) n must be of the form $12t+1$ or $36t+9$.

The bounds for $\sum_{p|n} (1/p)$ given in [1] have been improved in [3] as

(a) $$\frac{\log 2}{5 \log \left(\frac{5}{4}\right)} < \sum_{p|n} \frac{1}{p} < \log 2 + \frac{1}{338},$$

if n is of the form $12t + 1$,

(b) $$\frac{1}{3} + \frac{\log \frac{4}{3}}{5 \log \frac{5}{4}} < \sum_{p|n} \frac{1}{p} < \log \frac{18}{13} + \frac{53}{150},$$

if n is of the form $36t + 9$.

The object of this paper is to further improve the bounds for $\sum_{p|n} (1/p)$.

The following Tables I and II give numerical values for the bounds obtained in [3] and the bounds obtained in this paper respectively.

TABLE I

	Lower bound	Upper bound	Difference
(a)	.621	.696	.075
(b)	.591	.679	.088

Received by the editors August 23, 1962.

It can be easily seen from Table II that (a) if n is of the form $12t+1$, $.644 < \sum_{p|n} (1/p) < .693$, which is of range .049, a one-third cut in the length of the interval of [3] and (b) if n is of the form $36t+9$, $.596 < \sum_{p|n} (1/p) < .674$, which is of range .078, an improvement over [3] of about 12 per cent.

TABLE II

(α)	.644	.679	.035
(β)	.657	.693	.036
(γ)	.596	.674	.078
(δ)	.600	.662	.062

The bounds obtained are given by the following:

THEOREM. (α) *If n is of the form $12t+1$ and $5|n$,*

$$\frac{1}{5}+\frac{1}{7}+\frac{\log\frac{48}{35}}{11\log\frac{11}{10}} < \sum_{p|n}\frac{1}{p} < \frac{1}{5}+\frac{1}{2738}+\log\frac{50}{31}.$$

(β) *If n is of the form $12t+1$ and $5\nmid n$,*

$$\frac{1}{7}+\frac{\log\frac{12}{7}}{11\log\frac{11}{10}} < \sum_{p|n}\frac{1}{p} < \log 2.$$

(γ) *If n is of the form $36t+9$ and $5|n$,*

$$\frac{1}{3}+\frac{1}{5}+\frac{\log\frac{16}{15}}{17\log\frac{17}{16}} < \sum_{p|n}\frac{1}{p} < \frac{1}{3}+\frac{1}{5}+\frac{1}{13}+\log\frac{65}{61}.$$

(δ) *If n is of the form $36t+9$ and $5\nmid n$,*

$$\frac{1}{3}+\frac{\log\frac{4}{3}}{7\log\frac{7}{6}}<\sum_{p|n}\frac{1}{p}$$

$$<\frac{1}{3}+\frac{1}{338}+\log\frac{18}{13}.$$

PROOF. We prove this theorem in various cases and in each case one can see that either the lower bound or the upper bound for $\sum_{p|n}(1/p)$ as stated in this theorem is further improved.

Euler proved that n must be of the form $p_0^{\alpha_0}\cdot x^2$, where p_0 is a prime of the form $4\lambda+1$, α_0 is of the form $4\mu+1$, $x>1$ and $(p_0, x)=1$. Hence we can write $n=p_0^{\alpha_0}\cdot p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where α_r is even for $1\leqq r\leqq k$. We shall suppose as we may do without loss of generality that $p_1<p_2<\cdots<p_k$. Let $\sigma(n)$ denote the sum of the positive divisors of n. Since n is a perfect number, we have $\sigma(n)=2n$, from which it can easily be seen that

$$2\prod_{r=0}^{k}\left(1-\frac{1}{p_r}\right)=\prod_{r=0}^{k}\left(1-\frac{1}{p_r^{\alpha_r+1}}\right) \tag{A}$$
$$<1.$$

Therefore

$$\log 2<-\sum_{r=0}^{k}\log\left(1-\frac{1}{p_r}\right) \tag{B}$$
$$=\sum_{r=0}^{k}\frac{1}{p_r}+\frac{1}{2}\sum_{r=0}^{k}\frac{1}{p_r^2}+\frac{1}{3}\sum_{r=0}^{k}\frac{1}{p_r^3}+\cdots.$$

Taking logarithms of both sides of (A) and expressing them in series, we have

$$\log 2=\sum_{r=0}^{k}\sum_{i=1}^{\infty}\left[\frac{1}{ip_r^i}-\frac{1}{ip_r^{(\alpha_r+1)i}}\right] \tag{C}$$
$$=\sum_{r=0}^{k}\frac{1}{p_r}+\sum_{r=0}^{k}\sum_{i=1}^{\infty}\left[\frac{1}{(i+1)p_r^{i+1}}-\frac{1}{ip_r^{(\alpha_r+1)i}}\right].$$

(a) Suppose n is of the form $12t+1$. In this case it has been proved in [3, p. 134] that p_0 is of the form $12N+1$ and hence $p_0\geqq 13$.

(a_1) If $5|n$ and $7|n$, then $p_1=5$, $p_2=7$ and $p_r \geqq 11$ for $3 \leqq r \leqq k$. Now $\alpha_2 \geqq 4$ for, if $\alpha_2=2$, then $\sigma(p_2^{\alpha_2})=3.19$ and since $\sigma(n)=2n$ it would follow that $3|n$, which cannot hold.

From (B), we get that

$$\begin{aligned}
\log 2 < &-\log\left(1-\frac{1}{5}\right)-\log\left(1-\frac{1}{7}\right)+\frac{1}{p_0}+\frac{1}{2}\cdot\frac{1}{11}\cdot\frac{1}{p_0} \\
&+\frac{1}{3}\cdot\frac{1}{11^2}\cdot\frac{1}{p_0}+\cdots \\
&+\sum_{r=3}^{k}\frac{1}{p_r}+\frac{1}{2}\cdot\frac{1}{11}\cdot\sum_{r=3}^{k}\frac{1}{p_r} \\
&+\frac{1}{3}\cdot\frac{1}{11^2}\cdot\sum_{r=3}^{k}\frac{1}{p_r}+\cdots \\
= &\log\frac{5}{4}+\log\frac{7}{6}+\left(\frac{1}{p_0}+\sum_{r=3}^{k}\frac{1}{p_r}\right) \\
&\cdot\left(1+\frac{1}{2}\cdot\frac{1}{11}+\frac{1}{3}\cdot\frac{1}{11^2}+\cdots\right) \\
= &\log\frac{5}{4}+\log\frac{7}{6} \\
&+11\log\frac{11}{10}\left[\sum_{r=0}^{k}\frac{1}{p_r}-\frac{1}{5}-\frac{1}{7}\right]
\end{aligned}$$

therefore

$$\sum_{r=0}^{k}\frac{1}{p_r}>\frac{1}{5}+\frac{1}{7}+\frac{\log\dfrac{48}{35}}{11\log\dfrac{11}{10}}\cdot \tag{a_{1L}}$$

Also from (C), we get that

$$\begin{aligned}
\log 2 = &\sum_{r=0}^{k}\frac{1}{p_r}+\sum_{r=1}^{k}\sum_{i=1}^{\infty}\left[\frac{1}{(i+1)p_r^{i+1}}-\frac{1}{ip_r^{(\alpha_r+1)i}}\right] \\
&+\left(\frac{1}{2p_0^2}-\frac{1}{p_0^{\alpha_0+1}}\right)+\sum_{i=2}^{\infty}\left[\frac{1}{(i+1)p_0^{i+1}}-\frac{1}{ip_0^{(\alpha_0+1)i}}\right].
\end{aligned} \tag{D}$$

Now each term in the brackets of the second summation is positive, since $\alpha_r \geqq 2$ for $r>0$, and hence the second sum is positive. Similarly the fourth sum is also positive, and $1/2p_0^2 - 1/p_0^{\alpha_0+1} \geqq -1/2p_0^2 \geqq -1/338$, since $\alpha_0 \geqq 1$ and $p_0 \geqq 13$. Therefore

$$\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} + \sum_{i=1}^{\infty}\left[\frac{1}{(i+1)5^{i+1}} - \frac{1}{i(5^3)^i}\right]$$
$$+ \sum_{i=1}^{\infty}\left[\frac{1}{(i+1)7^{i+1}} - \frac{1}{i(7^5)^i}\right] - \frac{1}{338}, \text{ since } \alpha_1 \geqq 2 \text{ and } \alpha_2 \geqq 4$$
$$= \sum_{r=0}^{k} \frac{1}{p_r} - \log\left(1 - \frac{1}{5}\right) - \frac{1}{5} + \log\left(1 - \frac{1}{5^3}\right)$$
$$- \log\left(1 - \frac{1}{7}\right) - \frac{1}{7} + \log\left(1 - \frac{1}{7^5}\right) - \frac{1}{338}.$$

Therefore

$$(a_{1R}) \qquad \sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{5} + \frac{1}{7} + \frac{1}{338} + \log\frac{50\cdot 2401}{31\cdot 2801} < \frac{1}{5} + \frac{1}{2738} + \log\frac{50}{31}.$$

Hence by (a_{1L}) and (a_{1R}), (α) follows in this case.

(a_2) If $5 \mid n$ and $7 \nmid n$, then $p_1 = 5$ and $p_r \geqq 11$ for $2 \leqq r \leqq k$. Since α_0 is odd $(1+p_0) \mid \sigma(p_0^{\alpha_0})$ and hence $(1+p_0)/2 \mid n$ since $\sigma(n) = 2n$. Now p_0 is not 13, since otherwise it would follow that $7 \mid n$, which is not the case. Since p_0 is of the form $12N+1$, $p_0 \geqq 37$.

From (B) as in (a_1) we can get that

$$\log 2 < \log\frac{5}{4} + 11 \log\frac{11}{10}\left[\frac{1}{p_0} + \sum_{r=2}^{k} \frac{1}{p_r}\right]$$
$$= \log\frac{5}{4} + 11 \log\frac{11}{10}\left[\sum_{r=0}^{k} \frac{1}{p_r} - \frac{1}{5}\right];$$

therefore

$$(a_{2L}) \qquad \sum_{r=0}^{k} \frac{1}{p_r} > \frac{1}{5} + \frac{\log\frac{8}{5}}{11\log\frac{11}{10}} > \frac{1}{5} + \frac{1}{7} + \frac{\log\frac{48}{35}}{11\log\frac{11}{10}}.$$

From (D), arguing in a similar way as in (a_1), we can get that

$$\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} + \sum_{i=1}^{\infty}\left[\frac{1}{(i+1)5^{i+1}} - \frac{1}{i(5^3)^i}\right] - \frac{1}{2(37)^2}$$

$$= \sum_{r=0}^{k} \frac{1}{p_r} - \log\left(1-\frac{1}{5}\right) - \frac{1}{5} + \log\left(1-\frac{1}{5^3}\right) - \frac{1}{2738}.$$

Therefore

$$(a_{2R}) \qquad \sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{5} + \frac{1}{2738} + \log\frac{50}{31}.$$

Hence by (a_{2L}) and (a_{2R}), (α) follows in this case.

Thus (α) is proved.

(a_3) If $5\nmid n$ and $7\mid n$, then $p_1=7$ and $p_r \geqq 11$ for $2\leqq r\leqq k$. Now $\alpha_1\geqq 4$ as we have seen in (a_1) that $\alpha_1\neq 2$. From (B) as in the case (a_1), we get that

$$\log 2 < \log\frac{7}{6} + 11\log\frac{11}{10}\left[\sum_{r=0}^{k}\frac{1}{p_r} - \frac{1}{7}\right].$$

Therefore

$$(a_{3L}) \qquad \sum_{r=0}^{k} \frac{1}{p_r} > \frac{1}{7} + \frac{\log\frac{12}{7}}{11\log\frac{11}{10}}.$$

From (D), arguing in a similar way as in (a_1), we get that

$$\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} - \log\left(1-\frac{1}{7}\right) - \frac{1}{7} + \log\left(1-\frac{1}{7^5}\right) - \frac{1}{338}.$$

Therefore

$$(a_{3R}) \qquad \sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{7} + \frac{1}{338} + \log\frac{4802}{2801} < \log 2.$$

Hence by (a_{3L}) and (a_{3R}), (β) follows in this case.

(a_4) If $5\nmid n$ and $7\nmid n$, then $p_r\geqq 11$ for $1\leqq r\leqq k$.

From (B) as in (a_1), we get that $\log 2 < 11\log 11/10\cdot\sum_{r=0}^{k} 1/p_r$.

Therefore

$$(\mathrm{a}_{4L})\qquad \sum_{r=0}^{k}\frac{1}{p_r}>\frac{\log 2}{11\log\frac{11}{10}}>\frac{1}{7}+\frac{\log\frac{12}{7}}{11\log\frac{11}{10}}.$$

Now as in (a_2), we see that $(1+p_0)/2\,|\,n$. Let π be any prime dividing $(1+p_0)/2$, then $\pi\,|\,n$ and hence $\pi=p_j$ for some j satisfying $1\leqq j\leqq k$.

From (D), arguing in a similar way as in (a_1), we see that

$$\begin{aligned}\log 2 &> \sum_{r=0}^{k}\frac{1}{p_r}+\sum_{i=1}^{\infty}\left[\frac{1}{(i+1)p_j^{i+1}}-\frac{1}{i(p_j^3)^i}\right]\\ &\quad+\left(\frac{1}{2p_0^2}-\frac{1}{p_0^2}\right),\qquad \text{since } \alpha_0\geqq 1 \text{ and } \alpha_j\geqq 2\\ &> \sum_{r=0}^{k}\frac{1}{p_r}+\left(\frac{1}{2p_j^2}-\frac{1}{p_j^3}\right)-\frac{1}{2p_0^2}\\ &> \sum_{r=0}^{k}\frac{1}{p_r},\qquad \text{since } 11\leqq p_j\leqq\frac{1+p_0}{2}.\end{aligned}$$

Therefore

$$(\mathrm{a}_{4R})\qquad \sum_{r=0}^{k}\frac{1}{p_r}<\log 2.$$

Hence by (a_{4L}) and (a_{4R}), (β) follows in this case.

Thus (β) is proved.

(b) Suppose n is of the form $36t+9$. Since $3\,|\,n$, $p_1=3$.

(b_1) If $5\,|\,n$, then $7\nmid n$ in virtue of the result that 3.5.7 does not divide n (proved in Kühnel [4]).

($\mathrm{b}_{1\cdot 1}$) If at least one of 11 and 13 divides n, then obviously

$$\sum_{r=0}^{k}\frac{1}{p_r}>\frac{1}{3}+\frac{1}{5}+\frac{1}{13}>\frac{1}{3}+\frac{1}{5}+\frac{\log\frac{16}{15}}{17\log\frac{17}{16}}.$$

Otherwise,

($\mathrm{b}_{1\cdot 2}$) $p_r\geqq 17$ for $2\leqq r\leqq k$, if $p_0=5$; or

($\mathrm{b}_{1\cdot 3}$) $p_r\geqq 17$ for $3\leqq r\leqq k$, if $p_2=5$. In this particular case p_0 is also $\geqq 17$, since $p_0\neq 5$ and p_0 is not 13, since we are in the case where neither 11 nor 13 divides n.

In both the cases ($b_{1\cdot 2}$) and ($b_{1\cdot 3}$), from (B) as in (a_1) we can get that

$$\log 2 < \log\frac{3}{2} + \log\frac{5}{4} + 17\log\frac{17}{16}\left[\sum_{r=0}^{k}\frac{1}{p_r} - \frac{1}{3} - \frac{1}{5}\right].$$

Hence in any case under (b_1), we have that

$$(b_{1L})\qquad \sum_{r=0}^{k}\frac{1}{p_r} > \frac{1}{3} + \frac{1}{5} + \frac{\log\frac{16}{15}}{17\log\frac{17}{16}}\,.$$

For the upper bound, the proof for the cases (1) $p_0 \neq 5$ and (2) $p_0 = 5$ and $\alpha_1 \geqq 4$ are omitted as they are similar to the previous proofs. In both these cases we easily verify that the bound obtained is less than the bound obtained for the case $p_0 = 5$ and $\alpha_1 = 2$.

For this case, since $\alpha_1 = 2$, $\sigma(p_1^{\alpha_1}) = 13$, so $13 \mid n$.

We then obtain from (D), arguing in a similar way as in (a_1),

$$\begin{aligned}
\log 2 &> \sum_{r=0}^{k}\frac{1}{p_r} + \sum_{i=1}^{\infty}\left[\frac{1}{(i+1)3^{i+1}} - \frac{1}{i(3^3)^i}\right] \\
&\quad + \sum_{i=1}^{\infty}\left[\frac{1}{(i+1)5^{i+1}} - \frac{1}{i(5^2)^i}\right] + \sum_{i=1}^{\infty}\left[\frac{1}{(i+1)13^{i+1}} - \frac{1}{i(13^3)^i}\right] \\
&= \sum_{r=0}^{k}\frac{1}{p_r} - \log\left(1-\frac{1}{3}\right) - \frac{1}{3} + \log\left(1-\frac{1}{3^3}\right) - \log\left(1-\frac{1}{5}\right) - \frac{1}{5} \\
&\quad + \log\left(1-\frac{1}{5^2}\right) - \log\left(1-\frac{1}{13}\right) - \frac{1}{13} + \log\left(1-\frac{1}{13^3}\right).
\end{aligned}$$

Therefore

$$(b_{1R})\qquad \sum_{r=0}^{k}\frac{1}{p_r} < \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log\frac{65}{61}\,.$$

Hence by (b_{1L}) and (b_{1R}), (γ) follows.

(b_2) If $5 \nmid n$, then $p_r \geqq 7$ for $2 \leqq r \leqq k$ and $p_0 \geqq 13$.

From (B) as in (a_1) we get that

$$\log 2 < \log\frac{3}{2} + 7\log\frac{7}{6}\left[\sum_{r=0}^{k}\frac{1}{p_r} - \frac{1}{3}\right];$$

therefore

$$(b_{2L})\qquad \sum_{r=0}^{k}\frac{1}{p_r} > \frac{1}{3} + \log\frac{4}{3}\Big/ 7\log\frac{7}{6}\,.$$

From (D), arguing in a similar way as in (a_1), we get that

$$\log 2 > \sum_{r=0}^{k} \frac{1}{p_r} - \log\left(1 - \frac{1}{3}\right) - \frac{1}{3} + \log\left(1 - \frac{1}{3^3}\right) - \frac{1}{338}.$$

Therefore

$$(b_{2R}) \qquad \sum_{r=0}^{k} \frac{1}{p_r} < \frac{1}{3} + \frac{1}{338} + \log\frac{18}{13}.$$

Hence by (b_{2L}) and (b_{2R}), (δ) follows.

Thus the proof of the theorem is complete.

References

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