A REMARK ON UNIVERSAL SIGMA INTEGRABILITY

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Our purpose is to answer the question of Leader [1, p. 234], by producing an example of a sigma algebra σ of subsets of a set X and a *universally σ-integrable* function f (i.e., $\int f du$ exists for every countably additive set function u on σ) which is not measurable with respect to σ.

We shall show that the sigma algebra σ of Borel subsets of the interval $[0, 1]$ admits nonmeasurable universally σ-integrable functions (R. E. Zink suggested that this Borel algebra might yield an example).

Lusin has shown [2] that, subject to the continuum hypothesis, there exists an uncountable subset V of $X=[0, 1]$ such that each perfect nowhere dense subset G of X contains at most an enumerable set of points of V. Let H be a subset of V that is not a Borel set. We shall now show that the characteristic function $\chi(H)$ of H is universally sigma integrable. To this end suppose u is a countably additive set function on σ. Then [5] $u=c+j$ where c is continuous and $j=\sum j_k$ where j_k is a two valued jump function, the range of k is at most enumerable, and the variation of j is the sum of the variations of the j_k's; moreover, since the variation of u is the variation of j plus the variation $\bar{c}$ of c, each of j, c, and $\bar{c}$ is countably additive on σ. Corresponding to each j_k there is [4, Theorem 27.1] a point x_k such that $j_k(E)=j_k(E\cap[x_k])$ for every Borel subset E of X. Hence $\int\chi(H)dj=\sum j_k(H\cap[x_k])$ and it suffices to show that there exists a Borel subset B of X such that $H\subset B$ and $\bar{c}(B)=0$ (then $\int\chi(H)dc=0$). There exists [3] a set $K=\bigcup_i F_i$, F_i closed for $i\geqq 1$, of the first category in X and a set L such that $\bar{c}(L)=0$ and $K\cup L=X$. For each positive integer i there exists a perfect set G_i and an at most enumerable set H_i such that $F_i=G_i\cup H_i$. Thus $H\cap K=H\cap(\bigcup_i F_i)$ $=H\cap(\bigcup_i (G_i\cup H_i))=\bigcup_i (H\cap(G_i\cup H_i))\subset\bigcup_i((H\cap G_i)\cup H_i)$ is at most enumerable since every $H\cap G_i$ (F_i is a closed subset of X and K is of the first category in X; hence F_i is nowhere dense in X which implies that G_i is a perfect nowhere dense subset of X) and every H_i is at most enumerable. Since $\bar{c}$ is countably additive on σ and $\bar{c}$ vanishes on one point sets, $\bar{c}$ vanishes on countable sets. Hence $H=H\cap(K\cup L)\subset(H\cap K)\cup L$ and $\bar{c}(H\cap K)\cup L)\leqq\bar{c}(H\cap K)+\bar{c}(L)$ $=0$.

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