ESTIMATES ON THE STABILITY INTERVALS FOR HILL'S EQUATION

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The purpose of this note is to state and prove the following theorem.

THEOREM. *Consider the Hill's equation*

$$y'' + [\lambda + q(t)]y = 0, \tag{1}$$

where $q(t+\pi) = q(t)$, *and* $q(t)$ *is bounded. Let* λ_i *denote the ith eigenvalue corresponding to which a solution of* (1) *has period* π, *and* λ_i' *those corresponding to which* (1) *has a solution of period* 2π. *It is well known* [1] *that for these eigenvalues*

$$\lambda_0 < \lambda_1' \leqq \lambda_2' < \lambda_1 \leqq \lambda_2 < \lambda_3' \leqq \lambda_4' < \lambda_3 \leqq \lambda_4 < \cdots$$

and for all λ *in the intervals*

$$(\lambda_0, \lambda_1'), (\lambda_2', \lambda_1), (\lambda_2, \lambda_3'), (\lambda_4', \lambda_3), \cdots \tag{2}$$

has only bounded solutions. For λ *outside those intervals unbounded solutions occur. These intervals* $(-\infty, \lambda_0)$, (λ_1', λ_2'), (λ_1, λ_2), $(\lambda_3', \lambda_4') \cdots$ *are known as the instability intervals. Then if* $q(t)$ *has m continuous derivatives it follows that*

$$\lambda_{2k} - \lambda_{2k-1} = o\left(\frac{1}{k^{m-1}}\right)$$

as $k \to \infty$, *and similarly for* $\lambda_{2k}' - \lambda_{2k-1}'$.

This result was previously proved in [2] for $q(t)$, which are even in t. The following proof lifts that restriction.

The proof is similar to the one given in [2]. The solutions of (1) can be represented by

$$y = A(t)\sin\phi(t), \qquad y' = (\lambda + q(t))^{1/2} A(t)\cos\phi(t).$$

From these it follows that

$$\phi'(t) = (\lambda + q(t))^{1/2} + \frac{q'(t)}{4(\lambda + q(t))}\sin 2\phi(t), \tag{3}$$

$$\frac{A'(t)}{A(t)} = -\frac{q'(t)\cos^2\phi(t)}{2(\lambda + q(t))}. \tag{4}$$

Received by the editors September 17, 1962.

For sufficiently large λ, A must be an exponential function and cannot vanish. For solutions of period π we require that

$$y(0) = y(\pi), \qquad y'(0) = y'(\pi).$$

From this one can conclude that

$$\phi(\pi) = \phi(0) + 2k\pi, \qquad A(\pi) = A(0).$$

Similarly for solutions of period 2π

$$y(0) = -y(\pi), \qquad y'(0) = -y'(\pi)$$

so that

$$\phi(\pi) = \phi(0) + (2k+1)\pi, \qquad A(\pi) = A(0).$$

From (3) and (4) we then obtain

$$k\pi = \int_0^\pi (\lambda + q(t))^{1/2} dt + \int_0^\pi \frac{q'(t)}{4(\lambda + q(t))} \sin 2\phi(t) dt, \tag{5}$$

$$0 = -\frac{1}{2}\int_0^\pi \frac{q'(t)\cos^2\phi(t)}{\lambda + q(t)} dt. \tag{6}$$

$\phi(0)$ and λ are unknown, but once $\phi(t)$ is determined from (3), they can be determined from (5) and (6). By virtue of the fact that only (2ϕ) occurs in (5) and (6), one will determine for every k two λ's and two $\phi(0)$'s. In case $q(t)$ is even one finds for $\phi(0)$ the solutions $\phi(0)=0$, $\phi(0)=\pi/2$. The two λ's will be close. Asymptotically (4) shows that

$$\lambda^{1/2} = k + O\left(\frac{1}{k}\right)$$

for large k. When $q(t)$ has m continuous derivatives it follows from (4), by integrations by parts, that

$$\int_0^\pi \frac{q'(t)}{4(\lambda + q(t))} \sin 2\phi(t) dt = o\left(\frac{1}{\lambda^{(m+1)/2}}\right) = o\left(\frac{1}{k^{m+1}}\right).$$

By taking the difference of the two equations of type (4) corresponding to the same value of k one finds that

$$\int_0^\pi (\lambda_{2k} + q(t))^{1/2} dt - \int_0^\pi (\lambda_{2k-1} + q(t))^{1/2} dt = o\left(\frac{1}{k^{m+1}}\right)$$

and an application of the mean value theorem shows that

$$\lambda_{2k} - \lambda_{2k-1} = o\left(\frac{1}{k^{m-1}}\right)$$

which proves the theorem.

References

1. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

2. H. Hochstadt, *Asymptotic estimates for the Sturm-Liouville spectrum*, Comm. Pure Appl. Math. 14 (1961).

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A CLASS OF INTEGRAL EQUATIONS INVOLVING ULTRASPHERICAL POLYNOMIALS AS KERNEL

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Introduction. In deriving a solution of a certain aerodynamical problem, Ta Li [1] was led to a general class of integral equations, each of which has, as its kernel, a Chebyshev polynomial of first kind divided by the square root of the difference of two squares. Ta Li has obtained an exact solution of each of these singular integral equations. The solution is given in the form of a singular integral involving Chebyshev polynomial of first kind.

In this note the author obtains an inversion formula for a singular integral transform involving ultraspherical polynomials.

Ultraspherical polynomials. The ultraspherical polynomial of nth degree is denoted by $C_n^\lambda(x)$. It is defined as the polynomial solution of the differential equation [2]

$$(x^2-1)y''(x)+(2\lambda+1)xy'(x)-n(n+2\lambda)y(x)=0, \tag{1}$$

with the initial condition $y(1)=1$. Writing

$$y(x)=x^\delta z(x^2-1), \tag{2}$$

where

$$\delta=\begin{cases}0 & \text{when } n \text{ is even,}\\ 1 & \text{when } n \text{ is odd,}\end{cases} \tag{3}$$

we find, after letting

$$t=x^2-1, \tag{4}$$

the following differential equation for $z(t)$:

Received by the editors September 9, 1962.