which proves the theorem.

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A CLASS OF INTEGRAL EQUATIONS INVOLVING ULTRASPHERICAL POLYNOMIALS AS KERNEL

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Introduction. In deriving a solution of a certain aerodynamical problem, Ta Li [1] was led to a general class of integral equations, each of which has, as its kernel, a Chebyshev polynomial of first kind divided by the square root of the difference of two squares. Ta Li has obtained an exact solution of each of these singular integral equations. The solution is given in the form of a singular integral involving Chebyshev polynomial of first kind.

In this note the author obtains an inversion formula for a singular integral transform involving ultraspherical polynomials.

Ultraspherical polynomials. The ultraspherical polynomial of nth degree is denoted by $C_n^\lambda(x)$. It is defined as the polynomial solution of the differential equation [2]

$$(x^2 - 1)y''(x) + (2\lambda + 1)xy'(x) - n(n + 2\lambda)y(x) = 0, \tag{1}$$

with the initial condition $y(1) = 1$. Writing

$$y(x) = x^\delta z(x^2 - 1), \tag{2}$$

where

$$\delta = \begin{cases} 0 & \text{when } n \text{ is even,} \\ 1 & \text{when } n \text{ is odd,} \end{cases} \tag{3}$$

we find, after letting

$$t = x^2 - 1, \tag{4}$$

the following differential equation for $z(t)$:

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$$4t(t+1)z'' + \{2t(\delta+\lambda+1)+2\lambda+1\}z' - 4\left[\frac{n}{2}\right]\left\{\left[\frac{n+1}{2}\right]+\lambda\right\}z = 0, \tag{5}$$

where $[\xi]$ denotes the positive integral part of ξ. The polynomial solution of (5) satisfying $z(0)=1$ is

$$(t+1)^{-\delta/2}C_n^\lambda\{(t+1)^{1/2}\} = \frac{\left[\frac{n}{2}\right]!\Gamma(\lambda+1/2)2^{2\lambda}}{\left(\left[\frac{n-1}{2}\right]+\lambda\right)!\Gamma(1/2)} \times \sum_{k=0}^{[n/2]} \frac{2^{2k}\Gamma\left(\left[\frac{n-1}{2}\right]+k+\lambda+1\right)\Gamma(k+\lambda+1)}{\Gamma\left(\left[\frac{n}{2}\right]-k+1\right)\Gamma(2k+2\lambda+1)}\frac{t^k}{(k)!}. \tag{6}$$

Replacing t by x^2-1 in (6), one finds the following lemma:

LEMMA 1. *The ultraphserical polynomials $C_n^\lambda(x)$ defined by the differential equation* (1) *and the condition $y(1)=1$ can be written in the form*

$$C_n^\lambda(x) = \frac{\left[\frac{n}{2}\right]!\Gamma(\lambda+1/2)2^{2\lambda}}{\left(\left[\frac{n-1}{2}\right]+\lambda\right)!\Gamma(1/2)}\cdot x^\delta \times \sum_{k=0}^{[n/2]} \frac{2^{2k}\left(\left[\frac{n-1}{2}\right]+k+\lambda\right)!(k+\lambda)!}{\left(\left[\frac{n}{2}\right]-k\right)!(2k+2\lambda)!}\frac{(x^2-1)^k}{(k)!}, \tag{7}$$

where δ is given by (3).

(7) is the expression of the ultraspherical polynomials most suitable for our discussion.

Integral equations and their solutions. Consider the integral equations

$$\int_\sigma^1 \frac{C_n^\lambda(u/\sigma)y_n(u)}{(u^2-\sigma^2)^{1/2-\lambda}}du = f_n(\sigma), \qquad \sigma \in I,\ n = 1, 2, 3, \cdots, \tag{8}$$

where the integral is taken in the Riemann sense, $I=\{\sigma: c\leqq\sigma\leqq 1\}$, $c>0$ is a constant, and $f_n(\sigma)$ is defined on I. It is assumed that

(a) $-\frac{1}{2}<\lambda<\frac{1}{2}$,

(b) $f_n(1)=0$,

(c) $(d/d\sigma)[\sigma^n f_n(\sigma)]$ is piecewise continuous on I.

Condition (b) is necessary for $y_n(u)$ to remain finite on I. The same condition implies that $(d/d\sigma)[\sigma^n f_n(\sigma)]\not\equiv 0$ for otherwise $f_n(\sigma)=C\cdot\sigma^{-n}$, C is a constant, which contradicts (b). We prove the following theorem.

THEOREM. *Given $f_n(\sigma)$ on I, satisfying the conditions* (b) *and* (c), *the solution of* (8) *is given by*

$$y_n(u) = -\frac{2\cos(\pi\lambda)}{\pi}\sigma^{-2\lambda} \times\int_u^1 \frac{C_{n-1}^{\lambda}(u/v)d\{v^n f_n(v)\}}{v^{n-2\lambda-1}(v^2-u^2)^{1/2+\lambda}\,{}_2F_1[n,-\lambda;1;1-v^2/\sigma^2]} \tag{9}$$

where $C_n^\lambda(x)$, $n=1,2,3,\cdots$; $-\frac{1}{2}<\lambda<\frac{1}{2}$, are ultraspherical polynomials.

We want to note in passing that when $n=1$, the solution of

$$\int_\sigma^1 \frac{uy_1(u)du}{(u^2-\sigma^2)^{1/2-\lambda}} = \sigma f_1(\sigma),$$

as obtained by the method given in [3] under the condition (a) is

$$uy_1(u) = -\frac{2\cos(\pi\lambda)}{\pi}\frac{d}{du}\left[\int_u^1 \frac{v^2 f_1(v)}{(v^2-u^2)^{1/2+\lambda}}dv\right].$$

In the case $f_1(1)=0$, this can be reduced to the form

$$y_1(u) = -\frac{2\cos(\pi\lambda)}{\pi}\sigma^{-2\lambda}\int_u^1 \frac{C_0^{-\lambda}(u/v)d\{v^n f_n(v)\}}{v^{-2\lambda}(v^2-u^2)^{1/2+\lambda}\,{}_2F_1[1,-\lambda;1;1-v^2/\sigma^2]},$$

by writing

$$\frac{vdv}{(v^2-u^2)^{1/2+\lambda}} = \frac{1}{1-2\lambda}d\{(v^2-u^2)^{1/2-\lambda}\}$$

and by integrating by parts. The differentiation with respect to u is then carried out under the integral sign and by Kummer's transformation [4], we have

$$(v/\sigma)^{-2\lambda}\,{}_2F_1[1,-\lambda;1;1-v^2/\sigma^2] = 1.$$

It should be noted that condition (b) does not necessarily impose a restriction on $f_n(\sigma)$. This will be shown in a later section. If we take $\lambda=0$, we get the results obtained by Ta Li for Chebyshev polynomials.

As a preparation for the proof of the dual relation (8) and (9), we have to establish a summation formula. It can be easily shown that if $m\geqq 1$

$$\begin{aligned}&\frac{(m)!(m-1)!}{\Gamma(m+\lambda)\Gamma(m-\lambda)}\sum_{k=0}^{m}\sum_{\mu=0}^{m-1}\frac{(-)^{\mu}\Gamma(m+k+\lambda)\Gamma(m+\mu-\lambda)}{(m-k)!(m-\mu-1)!(k)!(\mu)!}\\ &\qquad\times\frac{(t-1)^{k+\mu}t^{-\mu}}{(k+\mu)!}\\ &\qquad= t^{1-m}\,{}_2F_1[-2m+1,\lambda+1;1;1-t],\end{aligned}\tag{10}$$

and

$$\begin{aligned}&\frac{(m)!(m)!}{\Gamma(m+\lambda+1)\Gamma(m-\lambda)}\sum_{k=0}^{m}\sum_{\mu=0}^{m}\frac{(-)^{\mu}\Gamma(m+k+\lambda+1)\Gamma(m+\mu-\lambda)}{(m-k)!(m-\mu)!(k)!(\mu)!}\\ &\qquad\times\frac{(t-1)^{k+\mu}t^{-\mu}}{(k+\mu)!}\\ &\qquad= t^{-m}\,{}_2F_1[-2m,\lambda+1;1;1-t],\end{aligned}\tag{11}$$

these results taken together can be written as

$$\begin{aligned}&\frac{\left[\frac{n}{2}\right]!\left[\frac{n-1}{2}\right]!}{\left(\left[\frac{n-1}{2}\right]+\lambda\right)!\left(\left[\frac{n-2}{2}\right]-\lambda\right)!}\\ &\qquad\times\sum_{k=0}^{[n/2]}\sum_{\mu=0}^{[n-1/2]}\frac{(-)^{\mu}\left(\left[\frac{n-1}{2}\right]+\lambda+k\right)!}{\left(\left[\frac{n}{2}\right]-k\right)!}\\ &\qquad\times\frac{\left(\left[\frac{n-2}{2}\right]+\mu-\lambda\right)!}{\left(\left[\frac{n-1}{2}\right]-\mu\right)!(k)!(\mu)!}\,\frac{(t-1)^{k+\mu}t^{-\mu}}{(k+\mu)!}\\ &\qquad= t^{-[n-1/2]}\,{}_2F_1\left[-\left[\frac{n}{2}\right]-\left[\frac{n-1}{2}\right],\lambda+1;1;1-t\right].\end{aligned}\tag{12}$$

By Kummer's transformation [4]

$$t^{-[n-1/2]}\,{}_2F_1\left[-\left[\frac{n}{2}\right]-\left[\frac{n-1}{2}\right],\ \lambda+1;1;1-t\right]$$

$$(13)\qquad = t^{[n/2]-\lambda}\,{}_2F_1\left[1+\left[\frac{n}{2}\right]+\left[\frac{n-2}{2}\right],\ -\lambda;1;1-t\right]$$

$$= t^{[n/2]-\lambda}\,{}_2F_1[n,\ -\lambda;\ 1;\ 1-t].$$

LEMMA 2. *Let n be an integer $\geqq 2$, then*

$$\frac{\left[\frac{n}{2}\right]!\left[\frac{n-1}{2}\right]!}{\Gamma\left(\left[\frac{n-1}{2}\right]+\lambda+1\right)\Gamma\left(\left[\frac{n-2}{2}\right]-\lambda+1\right)}$$

$$(14)\qquad \times\sum_{k=0}^{[n/2]}\ \sum_{\mu=0}^{[n-1/2]}\frac{(-)^{\mu}\Gamma\left(\left[\frac{n-1}{2}\right]+k+\lambda+1\right)}{\left(\left[\frac{n}{2}\right]-k\right)!}$$

$$\times\frac{\Gamma\left(\left[\frac{n-2}{2}\right]+\mu-\lambda+1\right)}{\left(\left[\frac{n-1}{2}\right]-\mu\right)!(k)!(\mu)!}\ \frac{(t-1)^{k+\mu}t^{-\mu}}{(k+\mu)!}$$

$$= t^{[n/2]-\lambda}\,{}_2F_1[n,\ -\lambda;\ 1;\ 1-t].$$

Proof of the dual relation. On the grounds of conditions (a), (b) and (c) it is obvious that the integral (9) exists, and that the double integral

$$(15)\qquad J = -\frac{2\cos(\pi\lambda)}{\lambda}\int_{\sigma}^{1}\frac{C_n^{\lambda}\left(\frac{u}{\sigma}\right)\sigma^{-2\lambda}}{(u^2-\sigma^2)^{1/2-\lambda}}\times\left(\int_u^1\frac{C_{n-1}^{-\lambda}\left(\frac{u}{v}\right)d\{v^n f_n(v)\}}{v^{n-2\lambda-1}(v^2-u^2)^{1/2+\lambda}\,{}_2F_1[n,\ -\lambda;\ 1;\ 1-v^2/\sigma^2]}\right)du$$

obtained by directly substituting (9) in (8), is convergent. This double integral can be written as

$$J = -\frac{2\cos(\pi\lambda)}{\pi}\lim_{\epsilon\to 0}\sigma^{-2\lambda}\int_{\sigma+\epsilon}^{1-\epsilon}\int_{u+\epsilon}^{1}\frac{C_n^{\lambda}\left(\frac{u}{\sigma}\right)C_{n-1}^{-\lambda}\left(\frac{u}{v}\right)d\{v^n f_n(v)\}}{v^{n-2\lambda-1}(u^2-\sigma^2)^{1/2-\lambda}}$$
$$\times\frac{du}{(v^2-u^2)^{1/2+\lambda}\,{}_2F_1[n,-\lambda;1;1-v^2/\sigma^2]}.$$

Since the integrals are uniformly bounded and there are at most a finite number of discontinuities of $(d/dv)(v^n f_n(v))$ in the region

$$R:\begin{cases}u+\epsilon\leqq v\leqq 1\\ \sigma+\epsilon\leqq u\leqq 1\end{cases}$$

it is justifiable to interchange the order of integration in R. Thus we obtain

$$J = -\frac{2\cos(\pi\lambda)}{\pi}\sigma^{-2\lambda}\lim_{\epsilon\to 0}\int_{\sigma+2\epsilon}^{1}\frac{d\{v^n f_n(v)\}}{v^{n-2\lambda-1}\,{}_2F_1[n,-\lambda;1;1-v^2/\sigma^2]}$$
$$\times\int_{\sigma+\epsilon}^{v-\epsilon}\frac{C_n^{\lambda}(u/\sigma)C_{n-1}^{-\lambda}(u/v)}{(u^2-\sigma^2)^{1/2-\lambda}(v^2-u^2)^{1/2+\lambda}}du.$$

Because $C_n^{\lambda}(u/\sigma)C_{n-1}^{-\lambda}(u/v)$ is continuous and finite in the interval $v\leqq u\leqq\sigma$ and

$$\int_{\sigma}^{v}\frac{du^2}{(u^2-\sigma^2)^{1/2-\lambda}(v^2-u^2)^{1/2+\lambda}} = \pi/\cos(\pi\lambda),$$

the integral

$$\int_{\sigma}^{v}\frac{C_n^{\lambda}\left(\frac{u}{\sigma}\right)C_{n-1}^{-\lambda}\left(\frac{u}{v}\right)}{(u^2-\sigma^2)^{1/2-\lambda}(v^2-u^2)^{1/2+\lambda}}du$$

exists, so that we can write

$$J = -\frac{2\cos(\pi\lambda)}{\pi}\sigma^{-2\lambda}\int_{\sigma}^{1}\frac{d\{v^n f_n(v)\}}{v^{n-2\lambda-1}\,{}_2F_1[n,-\lambda;1;1-(v^2/\sigma^2)]}\tag{16}$$
$$\times\int_{\sigma}^{v}\frac{C_n^{\lambda}\left(\frac{u}{\sigma}\right)C_{n-1}^{-\lambda}\left(\frac{u}{v}\right)}{(u^2-\sigma^2)^{1/2-\lambda}(v^2-u^2)^{1/2+\lambda}}du.$$

From Lemma 1 we have

$$
\begin{aligned}
&C_n^{\lambda}\left(\frac{u}{\sigma}\right)C_{n-1}^{-\lambda}\left(\frac{u}{v}\right)\\
&= \frac{\left[\frac{n}{2}\right]!\left[\frac{n-1}{2}\right]!\Gamma(1/2+\lambda)\Gamma(1/2-\lambda)}{\left(\left[\frac{n-1}{2}\right]+\lambda\right)!\left(\left[\frac{n-2}{2}\right]-\lambda\right)!\pi}\left(\frac{u}{v}\right)\left(\frac{u}{\sigma}\right)^{\delta}\\
&\times \sum_{k=0}^{[n/2]}\sum_{\mu=0}^{[n-1/2]}\frac{(-)^{\mu}2^{2(k+\mu)}\left(\left[\frac{n-1}{2}\right]+k+\lambda\right)!\left(\left[\frac{n-2}{2}\right]+\mu+\lambda\right)!}{\left(\left[\frac{n}{2}\right]-k\right)!\left(\left[\frac{n-1}{2}\right]-\mu\right)!(2\mu-2\lambda)!(2k+2\lambda)!}\\
&\times\frac{(k+\lambda)!(\mu-\lambda)!}{(k)!(\mu)!}\,\frac{(u^2-\sigma^2)^k(v^2-u^2)^{\mu}}{\sigma^{2k}\cdot v^{2\mu}}.
\end{aligned}
\tag{17}
$$

Substituting (17) in (16), writing $(u^2-\sigma^2)=(v^2-\sigma^2)x$ and making use of the following Euler's integral of the first kind

$$
\begin{aligned}
&\int_{\sigma}^{v}(u^2-\sigma^2)^{k+\lambda-1/2}(v^2-u^2)^{\mu-\lambda-1/2}du^2\\
&\qquad=\frac{\pi(2k+2\lambda)!(2\mu-2\lambda)!}{2^{2k+2\mu}(k+\lambda)!(\mu-\lambda)!}\,\frac{(v^2-\sigma^2)^{\mu+k}}{(k+\mu)!},
\end{aligned}
\tag{18}
$$

(16) reduces to

$$
J=-\int_{\sigma}^{1}\frac{\sigma^{-2\lambda}d\{v^n f_n(v)\}}{v^{n-2\lambda}\,{}_2F_1[n,-\lambda;1;1-v^2/\sigma^2]}\,(v/\sigma)^{\delta}B(t),
\tag{19}
$$

where $t=v^2/\sigma^2$ and

$$
\begin{aligned}
B(t)&=\frac{\left[\frac{n}{2}\right]!\left[\frac{n-1}{2}\right]!}{\left(\left[\frac{n-1}{2}\right]+\lambda\right)!\left(\left[\frac{n-2}{2}\right]-\lambda\right)!}\\
&\times\sum_{k=0}^{[n/2]}\sum_{\mu=0}^{[n-1/2]}\frac{(-)^{\mu}\left(\left[\frac{n-1}{2}\right]+k+\lambda\right)!\left(\left[\frac{n-2}{2}\right]+\mu-\lambda\right)!}{\left(\left[\frac{n}{2}\right]-k\right)!\left(\left[\frac{n-1}{2}\right]-\mu\right)!(k)!(\mu)!}\\
&\times\frac{(t-1)^{k+\mu}t^{-\mu}}{(k+\mu)!}.
\end{aligned}
\tag{20}
$$

By Lemma 2, we have

$$B(t) = t^{[n/2]-\lambda}\ {}_2F_1[n, -\lambda; 1; 1 - t]. \tag{21}$$

Introducing (21) into (19) and using the relation

$$2\left[\frac{n}{2}\right] + \delta = n, \tag{22}$$

one finds for $n \geqq 2$

$$J = -\sigma^{-n} \int_\sigma^1 d\{v^n f_n(v)\} = f_n(\sigma).$$

The case $n=1$ can be verified by direct substitution of (9) in (8). This completes the proof of the theorem.

The case $f_n(1) \neq 0$. To remove the restriction $f_n(1)=0$, one sets

$$f_n(\sigma) = \sigma^{-n} f_n(1) + \overset{*}{f_n}(\sigma), \tag{23}$$

and

$$y_n(u) = z_n(u) + w_n(u), \tag{24}$$

in the equation (8) and obtains

$$\int_\sigma^1 \frac{\overset{\lambda}{C_n}(u/\sigma) z_n(u)}{(u^2 - \sigma^2)^{1/2-\lambda}}\, du = \sigma^{-n} f_n(1), \tag{25}$$

and

$$\int_\sigma^1 \frac{\overset{\lambda}{C_n}(u/\sigma) w_n(u)}{(u^2 - \sigma^2)^{1/2-\lambda}}\, du = \overset{*}{f_n}(\sigma). \tag{26}$$

The solution of (25) is found to be

$$z_n(u) = \frac{2 f_n(1) \cos(\pi\lambda) \sigma^{-2\lambda} \overset{-\lambda}{C_{n-1}}(u)}{\pi\ {}_2F_1[n, -\lambda; 1; 1 - \sigma^{-2}](1 - u^2)^{1/2+\lambda}} \tag{27}$$

by Lemmas 1 and 2, while that of (27) is given in (9) as

$$\begin{aligned} w_n(u) = &- \frac{2\cos(\pi\lambda)}{\pi} \sigma^{-2\lambda} \\ &\times \int_u^1 \frac{\overset{-\lambda}{C_{n-1}}(u/v)\, d\{v^n \overset{*}{f_n}(v)\}}{v^{n-2\lambda-1}(v^2 - u^2)^{1/2+\lambda}\ {}_2F_1[n, -\lambda; 1; 1 - v^2/\sigma^2]} \cdot \end{aligned} \tag{28}$$

Consequently, the solution of (8), in this case, is

$$y_n(u) = \frac{2\cos(\pi\lambda)}{\pi}\sigma^{-2\lambda}\Bigg[\frac{f_n(1)C_{n-1}^{-\lambda}(u)}{(1-u^2)^{1/2+\lambda}\,{}_2F_1[n,-\lambda;1;1-\sigma^{-2}]} - \int_u^1 \frac{C_{n-1}^{-\lambda}(u/v)\,d\{u^n f_n^*(v)\}}{v^{n-2\lambda-1}(v^2-u^2)^{1/2+\lambda}\,{}_2F_1[n,-\lambda;1;1-v^2/\sigma^2]}\Bigg]. \tag{29}$$

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