

A SPECTRAL MAPPING THEOREM FOR FUNCTIONS OF TWO COMMUTING LINEAR OPERATORS

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Let \mathfrak{X} be a complex Banach space and \mathfrak{X}^* the dual space of \mathfrak{X} . Let \mathfrak{A} and \mathfrak{A}^* be the Banach algebras of all endomorphisms of \mathfrak{X} and \mathfrak{X}^* , respectively; if $T \in \mathfrak{A}$ the adjoint $T^* \in \mathfrak{A}^*$. We use \mathfrak{B} to denote the Banach algebra of all endomorphisms of \mathfrak{A} considered as a Banach space. As in [3] we associate with any $U, V \in \mathfrak{A}$, the operators $U^+, V^- \in \mathfrak{B}$ defined by $U^+(X) = UX$ and $V^-(X) = XV$. Note that $U^+V^- = V^-U^+$. Let $f(\zeta_1, \zeta_2)$ be a single-valued function of two complex variables that is analytic in both variables in a domain that contains the product $\sigma(U^+) \times \sigma(V^-)$ of the spectra of the operators U^+ and V^- . Then $f(U^+, V^-)$ is given by (see J. Schwartz [4])

$$(1) \quad \left(\frac{1}{2\pi i} \right)^2 \int_{\Gamma_1} \int_{\Gamma_2} (\zeta_1 I^+ - U^+)^{-1} (\zeta_2 I^- - V^-)^{-1} f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2,$$

where Γ_1 and Γ_2 are suitable contours and I is the identity operator in \mathfrak{A} . The operator defined by (1) belongs to \mathfrak{B} .

In this note we determine the relationship among the spectra of $f(U^+, V^-)$, U^+ and V^- thus extending the well-known result of the case of a single variable. G. Lumer and M. Rosenblum [3] have paved the way for this result in their analysis covering the case $f(\zeta_1, \zeta_2) = \sum_{i=1}^n f_i(\zeta_1)g_i(\zeta_2)$. We also study point spectra and eigenvectors, and apply our results to the Fréchet derivative.

It is shown in [3] that $\sigma(U^+) = \sigma(U)$ and $\sigma(V^-) = \sigma(V)$. The relations for point spectra are given in the following lemma. Let $\sigma_p(T)$ denote the point spectrum of the operator T . For any $x \in \mathfrak{X}$ and $y^* \in \mathfrak{X}^*$ we define the operator $x \otimes y^* \in \mathfrak{A}$ by $(x \otimes y^*)z = y^*(z)x$.

LEMMA. *Suppose $U, V \in \mathfrak{A}$ and that $u \in \mathfrak{X}$ and $v^* \in \mathfrak{X}^*$ are eigenvectors of U and V^* with corresponding eigenvalues μ and ν , respectively. Then*

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- (a) $U^+V^-(u \otimes v^*) = \mu\nu(u \otimes v^*),$
- (b) $\sigma_p(U^+) = \sigma_p(U),$
- (c) $\sigma_p(V^-) = \sigma_p(V^*).$

PROOF. (a) follows from the fact that $U(u \otimes v^*)V = (Uu) \otimes (V^*v^*)$. Then (a) implies $\sigma_p(U) \subset \sigma_p(U^+)$ and $\sigma_p(V^*) \subset \sigma_p(V^-)$. Suppose that $\xi \in \sigma_p(U^+)$ and $\eta \in \sigma(V^-)$ with corresponding eigenvectors X and Y , respectively. Then there exist $a, b \in \mathfrak{X}$ such that $Xa \neq \theta, Yb \neq \theta$. Choose $y^* \in \mathfrak{X}^*$ with $y^*(Yb) = 1$. By direct verification we see that $U(Xa) = \xi(Xa)$ and $V^*(Y^*y^*) = \eta(Y^*y^*)$. Hence $\xi \in \sigma_p(U)$ and $\eta \in \sigma(V^*)$. This completes the proof.

If $f(\zeta)$ is a single-valued function that is analytic in a complex domain that contains $\sigma(U)$, it is shown in [3] that

- (2) $(f(U))^+ = f(U^+),$
- (3) $(f(U))^- = f(U^-).$

THEOREM. If $f(U^+, V^-)$ is defined by (1) then

- (4) $\sigma(f(U^+, V^-)) = f(\sigma(U), \sigma(V)).$

Furthermore, if u and v^* are eigenvectors of U and V^* with corresponding eigenvalues μ and ν , respectively, then $u \otimes v^*$ is an eigenvector of $f(U^+, V^-)$ corresponding to the eigenvalue $f(\mu, \nu)$.

PROOF. Let $\lambda \in \sigma(f(U^+, V^-))$ and suppose $\lambda \notin f(\sigma(U), \sigma(V))$. Then the function $h(\zeta_1, \zeta_2) = (f(\zeta_1, \zeta_2) - \lambda)^{-1}$ is defined for (U^+, V^-) . Also, $h(U^+, V^-)(f(U^+, V^-) - \lambda I^+) = I^+$ which contradicts the assumption that $\lambda \in \sigma(f(U^+, V^-))$. The proof of (4) is completed in the manner of G. Lumer and M. Rosenblum [3, Theorems 9 and 10] on observing that for $X \in \mathfrak{A}$

$$f(U^+, V^-)X = \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{\Gamma_2} (\zeta_1 I - U)^{-1} X (\zeta_2 I - V)^{-1} f(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2.$$

The second part of the theorem may be verified directly with the aid of (2), (3) and the lemma.

Relation (4) is an extension of a result of Lumer and Rosenblum [3, Theorem 10]; M. Hausner [1] obtained (4) in the case of matrix algebras.

A function f with domain and range in \mathfrak{A} is *analytic* at $X \in \mathfrak{A}$ if the Fréchet differential has the form

$$df(X, H) = g(X^+, X^-)H$$

where g has domain in $\mathfrak{B} \times \mathfrak{B}$ and range in \mathfrak{B} . As in [5] the derivative $f'(X) = df(X, \cdot) = g(X^+, X^-)$. Let $f(\zeta)$ be a single-valued function of

a complex variable that is analytic in a domain that contains $\sigma(X)$. Then $f(Z)$ is defined for $\|Z - X\|$ small and has a Fréchet derivative (see J. Schwartz [4])

$$f^1(X) = f(X^+, X^-)$$

where the right side is defined by (1) with

$$(5) \quad \begin{aligned} f(\xi_1, \xi_2) &= \frac{f(\xi_1) - f(\xi_2)}{\xi_1 - \xi_2} && \text{if } \xi_1 \neq \xi_2 \\ &= f'(\xi_1) && \text{if } \xi_1 = \xi_2. \end{aligned}$$

We then have the following corollary.

COROLLARY. *The spectrum of the Fréchet derivative considered as an operator on the Banach space \mathfrak{A} is given by*

$$(6) \quad \sigma(f^1(X)) = f(\sigma(X), \sigma(X))$$

where $f(\xi_1, \xi_2)$ is defined by (5). If x and y^* are eigenvectors of X and X^* corresponding to eigenvalues ξ and η , respectively, then $x \otimes y^*$ is an eigenvector of $f^1(X)$ corresponding to the eigenvalue $f(\xi, \eta)$.

M. Hausner [1] obtained (6) in the case of matrix algebras.

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