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NOTE ON A NONLINEAR VOLTERRA EQUATION

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1. **Introduction.** We investigate the solutions of

$$(1.1) \quad x'(t) = - \int_0^t a(t-\tau)g(x(\tau))d\tau \quad \left(' = \frac{d}{dt} \right)$$

as $t \rightarrow \infty$, where $a(t)$ is completely monotonic on $0 \leq t < \infty$ and where $g(x)$ is a (nonlinear) spring. Under this hypothesis, (1.1) was shown in [2] to be relevant to certain physical applications and results were obtained there for the linear case $g(x) \equiv x$. (If $a(t) \equiv a(0)$, then (1.1) reduces to the nonlinear oscillator $x'' + a(0)g(x) = 0$.) Equation (1.1) was studied in [1] under less hypothesis on $a(t)$. However, while the result is weaker than that of [1], the present approach draws together such different notions of positivity as Liapounov functions, completely monotonic functions, and kernels of positive type. It also provides a new Liapounov function for (1.1). Specifically, we prove the

THEOREM. *Let $a(t)$ and $g(x)$ satisfy*

$$(1.2) \quad a(t) \in C[0, \infty), (-1)^k a^{(k)}(t) \geq 0 \quad (0 < t < \infty; k = 0, 1, 2, \dots),$$

$$(1.3) \quad g(x) \in C(-\infty, \infty), xg(x) > 0 \quad (x \neq 0), G(x) = \int_0^x g(\xi)d\xi \rightarrow \infty \\ (|x| \rightarrow \infty).$$

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If $a(t) \neq a(0)$ and if $u(t)$ is any solution of (1.1) which exists on $0 \leq t < \infty$, then

$$(1.4) \quad \lim_{t \rightarrow \infty} u^{(j)}(t) = 0 \quad (j = 0, 1, 2).$$

In [1] only $k=0, 1, 2, 3$ is required in the analogue of (1.2), rather than complete monotonicity. The Liapounov function used there was

$$(1.5) \quad \begin{aligned} E(t) = & G(u(t)) + \frac{1}{2} a(t) \left[\int_0^t g(u(\tau)) d\tau \right]^2 \\ & - \frac{1}{2} \int_0^t a'(t-\tau) \left[\int_\tau^t g(u(s)) ds \right]^2 d\tau \geq 0. \end{aligned}$$

In (1.5) and the sequel $u(t)$ is the solution of (1.1) on $0 \leq t < \infty$ being considered. Remarks concerning existence, uniqueness, as well as background information and references, may be found in [1].

2. Positivity. In this section we motivate the hypothesis (1.2) and also obtain the Liapounov function. Suppose that the origin of the problem is such that $a(t) \geq 0$, $a(t) \in C[0, \infty)$, $a'(t) \in L_1(0, T)$ for each $0 < T < \infty$, and (1.3) are all required. Then clearly

$$(2.1) \quad V(t) = G(u(t)) + \frac{1}{2} \int_0^t \int_0^t a(\tau+s) g(u(t-\tau)) g(u(t-s)) d\tau ds$$

is nonnegative if the second term is. (From (1.1) and (1.3), $V(t)$ may be interpreted as the sum of a potential and kinetic energy.)

The second (or kinetic) term of $V(t)$ will be nonnegative if it is assumed that $a(\tau+s)$ is a kernel of positive type [3, p. 270] on the square $0 < \tau, s < t$ for each $0 < t < \infty$. By a theorem of Boas and Widder [3, pp. 273-275], this will be the case if and only if

$$(2.2) \quad a(t) = \int_{-\infty}^{\infty} \exp[-\xi t] d\alpha(\xi) \quad (0 < t < \infty),$$

where $\alpha(\xi)$ is nondecreasing on $-\infty < \xi < \infty$ (and may be assumed normalized: $\alpha(0) = 0$, $\alpha(\xi) = \frac{1}{2} [\alpha(\xi+) + \alpha(\xi-)]$).

Differentiating (2.1) yields

$$(2.3) \quad V'(t) = \int_0^t \int_0^t a'(\tau+s) g(u(t-\tau)) g(u(t-s)) d\tau ds,$$

where we have used the above assumptions, (1.1), the absolute continuity of $a(t)$ on $0 < \epsilon \leq t \leq T < \infty$ implied by (2.2), and an obvious change of variables. If $V(t)$ is to serve as a Liapounov function for

(1.1), then $V'(t)$ must be nonpositive. By (2.3) the latter will be assured if $-a'(\tau+s)$ is a kernel of positive type on the square $0 < \tau, s < t$ for each $0 < t < \infty$. However, this is compatible with (2.2) if and only if $\alpha(-\infty) = \alpha(0-)$. Thus, also using $a(t) \in C[0, \infty)$, one has

$$(2.4) \quad a(t) = \int_0^\infty \exp[-\xi t] d\alpha(\xi) \quad (0 \leq t < \infty),$$

where $\alpha(\infty) < \infty$. By a theorem of S. Bernstein [3, p. 160], (1.2) and (2.4) are equivalent.

Having motivated the hypothesis (1.2) and obtained a Liapounov function $V(t)$, it is interesting to compare the two Liapounov functions (1.5) and (2.1). It is obvious that if $a(t) \equiv a(0)$, then $E(t) \equiv V(t)$ ($0 \leq t < \infty$). We now prove a strong form of the converse statement. In particular, if $u(0) \neq 0$ and if $E(t) \equiv V(t)$ ($0 \leq t \leq t_0$) for some $0 < t_0 < \infty$, then $a(t) \equiv a(0)$ ($0 \leq t < \infty$). For this one need only assume that $a(t) \in C[0, \infty)$, $(-1)^k a^{(k)}(t) \geq 0$ ($0 < t < \infty$; $k=0, 1, 2$), $g(x) \in C(-\infty, \infty)$, and $g(x)=0$ implies $x=0$. Direct calculations show that (1.5) and (2.1) may also be written as

$$E(t) = G(u(t)) + \int_0^t g(u(\tau)) \left\{ \int_\tau^t a(t-\tau) g(u(s)) ds \right\} d\tau,$$

$$V(t) = G(u(t)) + \int_0^t g(u(\tau)) \left\{ \int_\tau^t a(2t-\tau-s) g(u(s)) ds \right\} d\tau.$$

Hence

$$(2.5) \quad \int_0^t g(u(\tau)) \left\{ \int_\tau^t [a(t-\tau) - a(2t-\tau-s)] g(u(s)) ds \right\} d\tau \equiv 0 \quad (0 \leq t \leq t_0).$$

From the hypothesis on $a(t)$ one has $a(t-\tau) - a(2t-\tau-s) \geq 0$ for $0 \leq \tau, s \leq t$. As $u(0) \neq 0$, there exists by continuity and the hypothesis on $g(x)$ a $0 < t_1 \leq t_0$ such that $g(u(t)) \neq 0$ for $0 \leq t \leq t_1$. It is now clear from (2.5) that $a(t-\tau) \equiv a(2t-\tau-s)$ ($0 \leq \tau \leq s \leq t \leq t_1$). The latter together with $a(t) \in C[0, \infty)$ easily implies $a(t) \equiv a(0)$ ($0 \leq t \leq 2t_1$). Hence $a'(t) \equiv 0$ ($0 \leq t \leq 2t_1$). However, as $-a'(t)$, $a''(t) \geq 0$ it follows that $a'(t) \equiv 0$ ($0 \leq t < \infty$) so that $a(t) \equiv a(0)$ ($0 \leq t < \infty$), which proves the assertion.

3. Proof of the Theorem. Let $a(t)$, $V(t)$ be given by (2.4), (2.1) respectively. Define

$$(3.1) \quad \Gamma(\xi, t) = \int_0^t \exp[-\xi(t-\tau)]g(u(\tau))d\tau \quad (0 \leq \xi, t < \infty).$$

Then

$$(3.2) \quad \frac{\partial \Gamma}{\partial t}(\xi, t) = g(u(t)) - \xi \Gamma(\xi, t) \quad (0 \leq \xi, t < \infty).$$

Clearly $\Gamma(\xi, t)$, $\Gamma_t(\xi, t)$ are bounded functions of ξ on $0 \leq \xi < \infty$ for each fixed t in $0 \leq t < \infty$. Hence, by Fubini's theorem, one may write (1.1) (with $u(t)$ replacing $x(t)$), (2.1) as

$$(3.3) \quad u'(t) = - \int_0^\infty \Gamma(\xi, t) d\alpha(\xi) \quad (0 \leq t < \infty),$$

$$(3.4) \quad V(t) = G(u(t)) + \frac{1}{2} \int_0^\infty \Gamma^2(\xi, t) d\alpha(\xi) \geq 0 \quad (0 \leq t < \infty)$$

respectively and, moreover,

$$(3.5) \quad V'(t) = - \int_0^\infty \xi \Gamma^2(\xi, t) d\alpha(\xi) \leq 0 \quad (0 \leq t < \infty).$$

From (3.4), (3.5) one has

$$G(u(t)) \leq V(t) \leq V(0) = G(u_0) \quad (0 \leq t < \infty),$$

where $u_0 = u(0)$. It follows from (1.3) that

$$(3.6) \quad |u(t)| \leq K < \infty \quad (0 \leq t < \infty).$$

In (3.6) and subsequent formulas $K = K(u_0) < \infty$, where K may vary from formula to formula, and $K(u_0) \rightarrow 0$ as $u_0 \rightarrow 0$. Thus

$$(3.7) \quad |\Gamma(\xi, t)| \leq Kt, \quad |\xi \Gamma(\xi, t)| \leq K, \quad |\Gamma_t(\xi, t)| \leq K \quad (0 \leq \xi, t < \infty).$$

Differentiating (3.3) (using Fubini's theorem) yields

$$(3.8) \quad u''(t) = -g(u(t))\alpha(\infty) + \int_0^\infty \xi \Gamma(\xi, t) d\alpha(\xi) \quad (0 \leq t < \infty),$$

which together with (1.3), (3.6), and (3.7) implies

$$(3.9) \quad |u''(t)| \leq K \quad (0 \leq t < \infty).$$

By (3.6), (3.9), and the mean value theorem one has

$$(3.10) \quad |u'(t)| \leq K \quad (0 \leq t < \infty).$$

From (3.2), (3.5) there results

$$V''(t) = -2g(u(t)) \int_0^\infty \xi \Gamma(\xi, t) d\alpha(\xi) + 2 \int_0^\infty \xi^2 \Gamma^2(\xi, t) d\alpha(\xi) \quad (0 \leq t < \infty),$$

so that by (1.3), (3.6), (3.7) one has $|V''(t)| \leq K$ on $0 \leq t < \infty$. The latter together with (3.4), (3.5), and the mean value theorem implies (see Lemma 1 of [1])

$$(3.11) \quad \lim_{t \rightarrow \infty} V'(t) = - \lim_{t \rightarrow \infty} \int_0^\infty \xi \Gamma^2(\xi, t) d\alpha(\xi) = 0.$$

We assert that there exists a $\xi_1 > 0$ such that $\Gamma(\xi_1, t) \rightarrow 0$ as $t \rightarrow \infty$. Suppose not and let $0 < \xi_0 < \infty$. Then there exist a $\lambda = \lambda(\xi_0) > 0$ and a sequence $\{t_n = t_n(\xi_0)\}$, where $t_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $|\Gamma(\xi_0, t_n)| \geq \lambda$. From (3.1) one has

$$(3.12) \quad \frac{\partial \Gamma}{\partial \xi}(\xi, t) = - \int_0^t \exp[-\xi(t-\tau)](t-\tau)g(u(\tau))d\tau \quad (0 \leq \xi, t < \infty).$$

Let $\delta = \min(\xi_0/2, \lambda \xi_0^2/8K_1)$, $I_{\xi_0} = \{\xi \mid |\xi - \xi_0| \leq \delta\}$, where $|g(u(t))| \leq K_1$ on $0 \leq t < \infty$. Then by the mean value theorem and (3.12) one obtains $|\Gamma(\xi, t_n) - \Gamma(\xi_0, t_n)| \leq \lambda/2$ ($\xi \in I_{\xi_0}$) so that, as $|\Gamma(\xi_0, t_n)| \geq \lambda$, $|\Gamma(\xi, t_n)| \geq \lambda/2$ ($\xi \in I_{\xi_0}$). Hence

$$\begin{aligned} \int_0^\infty \xi \Gamma^2(\xi, t_n) d\alpha(\xi) &\geq \int_{I_{\xi_0}} \xi \Gamma^2(\xi, t_n) d\alpha(\xi) \\ &\geq \frac{\lambda^2 \xi_0}{8} [\alpha(\xi_0 + \delta) - \alpha(\xi_0 - \delta)] \geq 0, \end{aligned}$$

which with (3.11) yields $\alpha(\xi_0 + \delta) = \alpha(\xi_0 - \delta)$. As this is true for each $0 < \xi_0 < \infty$, it follows that $\alpha(0+) = \alpha(\infty)$ which contradicts $a(t) \neq a(0)$. Thus, there exists a $\xi_1 > 0$ with the asserted property.

Let $f(t) = \exp[-\xi_1 t]$, $p(t) = g(u(t))$ for $0 \leq t < \infty$ and $f(t) = p(t) \equiv 0$ for $-\infty < t < 0$. From the preceding paragraph one has

$$(3.13) \quad \lim_{t \rightarrow \infty} \Gamma(\xi_1, t) = \lim_{t \rightarrow \infty} \int_{-\infty}^\infty f(t-\tau)p(\tau)d\tau = 0.$$

By applying Pitt's form of Wiener's tauberian theorem [3, p. 211] to (3.13), we now show that $p(t) \rightarrow 0$ as $t \rightarrow \infty$. (A longer elementary argument could also be used here.) Clearly, $f \in L_1(-\infty, \infty)$ and its Fourier transform $\hat{f}(s) = (2\pi)^{-1/2}(\xi_1 + is)^{-1} \neq 0$ for $-\infty < s < \infty$. As $|p(t)| \leq K$ ($-\infty < t < \infty$), there remains only to show that $p(t)$ is a slowly decreasing function in $(-\infty, \infty)$. For this it suffices to show

that $p(t_i) - p(s_i) \rightarrow 0$ as $i \rightarrow \infty$ if $\{t_i\}, \{s_i\}$ are any sequences satisfying $t_i > s_i > 0$ and $s_i \rightarrow \infty, t_i - s_i \rightarrow 0$ as $i \rightarrow \infty$. However, from (1.3), (3.6), (3.10), the mean value theorem, and uniform continuity, it is clear that $g(u(t_i)) - g(u(s_i)) \rightarrow 0$ as $i \rightarrow \infty$ for such sequences. Thus, $p(t) \rightarrow 0$ as $t \rightarrow \infty$, which together with (1.3) and (3.6) yields $u(t) \rightarrow 0$ as $t \rightarrow \infty$ (i.e., (1.4, $j=0$)).

From (1.4, $j=0$) and (3.9) one has (1.4, $j=1$) by the mean value theorem. From (1.4, $j=0$) and (3.1) it is an elementary exercise to show that $\xi\Gamma(\xi, t) \rightarrow 0$ as $t \rightarrow \infty$ uniformly with respect to ξ on $0 \leq \xi < \infty$. This together with (1.4, $j=0$), (1.3), and (3.8) implies (1.4, $j=2$), which completes the proof.

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