

THE UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS ON A TOPOLOGICAL SPACE

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Two questions motivate this study: To what extent is it possible to restructure the common domain of a family of continuous functions so that they all become uniformly continuous? When can a group of homeomorphisms of a completely regular space be interpreted as a topological transformation group? It happens that an answer to the first, in the case where range and domain coincide, provides the answer to the second.

The concepts and notation used in this paper, if nonstandard, are defined in [2], with the following exceptions: A structure V_I for a set X is called *open* if for each $x \in X$, $\alpha \in I$, and $y \in V_\alpha(x)$, there is a $\beta \in I$ such that $V_\beta(y) \subseteq V_\alpha(x)$. V_I is called *finite* if the space (X, V_I) is *totally bounded*: For each $\alpha \in I$ there is a finite $S \subseteq X$ such that $X = V_\alpha(S) = \bigcup_{x \in S} V_\alpha(x)$.

Since every topological space admits a structure [1], the following result implies that in any purely topological study of the family of all functions from one space to another, the notion of continuity can always be replaced by the notion of uniform continuity without loss of generality.

THEOREM 1. *Given a family F of continuous functions $f: X \rightarrow Y$ from a space (X, U_I) to a space (Y, V_J) , there exists a coarsest structure W_H as fine as U_I such that every member of F is uniformly continuous relative to the pair (W_H, V_J) . U_I and W_H define the same topology for X . If both U_I and V_J are symmetric, open, locally transitive, or transitive, then so is W_H , respectively. If both U_I and V_J are finite and V_J is uniform, then W_H is also finite.*

PROOF. Let K be the class of all finite subsets of F . Define

$$W_{\alpha a A} = U_\alpha \cap \bigcap_{g \in A} g^{-1} V_a g,$$

for $\alpha \in I$, $a \in J$, $A \in K$. $W_{I \times J \times K}$ is clearly reflexive and, since

$$[U_\alpha \cap U_\beta] \cap \bigcap_{g \in A \cup B} g^{-1} [V_a \cap V_b] g \subseteq W_{\alpha a A} \cap W_{\beta b B},$$

it is a base for a structure W_H . This structure is obviously as fine as U_I . On the other hand, U_I is locally as fine as W_H : Given $x \in X$ and

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$(\alpha, a, A) \in I \times J \times K$, it is possible to choose $\beta \in I$ so that $U_\beta \subseteq U_\alpha$ and so that $U_\beta(x) \subseteq g^{-1}V_ag(x)$, for all $g \in A$, because A is a finite set of continuous functions and U_I is closed under intersection. Then $U_\beta(x) \subseteq W_{aaA}(x)$. Thus U_I and W_H define the same topology. Now for each $f \in F$ and each $a \in J$, $W_{aa\{f\}} \subseteq f^{-1}V_af$, for any $\alpha \in I$, whence, f is uniformly continuous *re* (W_H, V_J) . Let us show that W_H is as coarse as possible. Suppose that each member of F is uniformly continuous *re* (W'_H, V_J) , for some W'_H as fine as U_I . Given $(\alpha, a, A) \in I \times J \times K$, choose $\alpha' \in H'$ so that $W'_{\alpha'} \subseteq g^{-1}V_ag$, for all $g \in A$, and so that $W'_{\alpha'} \subseteq U_\alpha$. Then $W'_{\alpha'} \subseteq W_{aaA}$. Thus W'_H is as fine as W_H . It is also straightforward to show that the various properties possessed by U_I and V_J are inherited (as recessive traits) by W_H . For example, suppose both U_I and V_J are locally transitive. Given $x \in X$ and $(\alpha, a, A) \in I \times J \times K$, choose $\beta \in I$ so that $U_\beta^2(x) \subseteq U_\alpha(x)$ and choose, for each $g \in A$, $b(g) \in J$ so that $V_{b(g)}^2(g(x)) \subseteq V_a(g(x))$. Then let $V_b = \bigcap_{g \in A} V_{b(g)}$. We have by a simple calculation with dyadic relations, $W_{\beta bA}(x) \subseteq U_\beta^2(x) \cap \bigcap_{g \in A} g^{-1}V^2g(x) \subseteq W_{aaA}(x)$, whence, W_H is also locally transitive. Finally, consider the hypothesis that U_I and V_J be finite. There is no difficulty in showing that subspaces of totally bounded uniform spaces are totally bounded, so if V_J is uniform, then the range of each member of F is totally bounded. Given $(\alpha, a, A) \in I \times J \times K$, we shall find a finite subset $S \subseteq X$ such that $X = W_{aaA}(S)$. Let $S_0 \subseteq X$ be a finite subset for which $X = U_\alpha(S_0)$ and, for each $g \in A$, let $T_g \subseteq g(X)$ be a finite subset for which $g(X) \subseteq V_a(T_g)$. Construct $S_g \subseteq X$ by choosing an $s \in g^{-1}(y)$, for each $y \in T_g$. Then it is easy to see that $S = S_0 \cup \bigcup_{g \in A} S_g$ has the desired properties. Therefore W_H is also finite.

Query. Are there subspaces of totally bounded spaces with symmetric locally transitive structures which are not totally bounded?

COROLLARY (LEVINE). *If U_I and V_J are equivalent to metric structures and F is countable, then W_H is equivalent to a metric structure.*

PROOF. A (Hausdorff) structure is equivalent to a metric structure if and only if it is uniform and has a countable base [6]. If F is countable and countable bases are involved in the construction of W_H , then W_H receives a countable base.

Note. In [3] an appropriate metric for X is given explicitly in terms of F and the original metrics for X and Y . In fact, if the original metrics are δ_1 and δ_2 , respectively, and δ_1^* is defined on $X \times X$ by

$$\delta_1^*(p, q) = \delta_1(p, q) + \sum_{i=1}^{\infty} 2^{-i} \min[1, \delta_2(f_i(p), f_i(q))],$$

where $F = \{f_1, f_2, \dots\}$, then δ_1 and δ_1^* define the same topology for X and every member of F is (δ_1^*, δ_2) -uniformly continuous. Note that δ_2 need not be changed. (See also [4].)

THEOREM 2. *Given a space (X, U_I) and a family F of continuous transformations $f: X \rightarrow X$, there is a coarsest structure W_H as fine as U_I , relative to which every member of F is uniformly continuous. U_I and W_H define the same topology for X and W_H inherits as in Theorem 1 *mutatis mutandis* the structure properties possessed by U_I .*

PROOF. Let F^* be the semigroup generated by F under composition, plus the identity $\Delta: X \rightarrow X: x \rightarrow x$. Let K be the class of all finite subsets of F^* containing Δ . Define

$$W_{\alpha A} = \bigcap_{g \in A} g^{-1} U_{\alpha} g,$$

for $\alpha \in I$, $A \in K$. The proof then proceeds very much like that of Theorem 1, except that we show uniform continuity as follows: Given $f \in F$ and $(\alpha, A) \in I \times K$, let $B = \{gf\}_{g \in A} \cup \{\Delta\}$. Then $B \in K$ and $W_{\alpha B} = U_{\alpha} \cap \bigcap_{g \in A} f^{-1} g^{-1} U_{\alpha} g f \subseteq f^{-1} W_{\alpha A} f$, so that f is uniformly continuous *re* W_H .

COROLLARY. *Every group of homeomorphisms of a completely regular space is a topological homeomorphism group. That is, there exists an admissible topology relative to which composition and inversion are continuous.*

PROOF. Let the topology of the given completely regular space be defined by a uniform structure V_I relative to which every continuous $f: X \rightarrow X$ is uniformly continuous. The existence of such a V_I follows from a well-known theorem by Weil [6] together with Theorem 2. Define for the set F of homeomorphisms in question the uniform structure V_I^* by putting $V_{\alpha}^*(f) = \{g \in F: g \subseteq V_{\alpha} f\}$ ($\alpha \in I, f \in F$). This provides F with the "topology of uniform convergence," with respect to which the map $F \times X \rightarrow X: (f, x) \rightarrow f(x)$ is continuous, as is well known. The continuity of $\Phi: F \times F \rightarrow F: (f, g) \rightarrow f^{-1}g$ follows from a theorem proved by Ford [5] together with the uniform continuity of the members of F . In fact, given $f, g \in F$ and $\alpha \in I$, choose $\beta \in I$ so that $V_{\beta}^2 \subseteq V_{\alpha}$ and $\gamma \in I$ so that V_{γ} is symmetric and $f^{-1}V_{\gamma} \subseteq V_{\beta} f^{-1}$. Then Φ maps $V_{\gamma}^*(f) \times V_{\gamma}^*(g)$ into $V_{\alpha}^*(f^{-1}g)$.

Note. There is also a corollary to Theorem 2 similar to the one following Theorem 1. In fact, if $F = \{f_1, f_2, \dots\}$ is a countable family of continuous transformations of a metric space (X, δ) , then the function $\delta^*: X \times X \rightarrow [0, \infty)$ defined by

$$\delta^*(p, q) = \delta(p, q) + \sum_{r=1}^{\infty} \sum_{\kappa \in S_r} \left(\frac{1}{2}\right)^{k_1 + \dots + k_r} \delta'(f_{k_1} \dots f_{k_r}(p), f_{k_1} \dots f_{k_r}(q)),$$

where $\delta' = \min [1, \delta]$ and S_r is the set of all r -tuples $\kappa = (k_1, \dots, k_r)$ of natural numbers, can be seen to be a metric defining the same topology as δ and such that, for every $k = 1, 2, \dots$, and all $p, q \in X$, $\delta^*(f_k(p), f_k(q)) \leq 2^k \delta^*(p, q)$. (Cf. [4].)

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